Chapter 18. Integration Over General Measure Spaces

Note. We now define integrals of extended real-valued functions over measure spaces much in the same way we did for Lebesgue integrals. Some of the details are different—for example we prove Fatou's Lemma early on and other results, such as the Monotone Convergence Theorem follow later.

Section 18.1. Measurable Functions

Note. In this section, we define and prove some of the familiar properties of measurable functions (such as the Simple Approximation Theorem) in the setting of extended real-valued functions defined on measure spaces.

Proposition 18.1. Let $(X, \mathcal{M})$ be a measure space and f an extended real-valued function defined on X. The following are equivalent:

(i) For each $c \in \mathbb{R}$, the set $\{x \in X \mid f(x) < c\}$ is measurable (i.e., in $\mathcal{M}$).

(ii) For each $c \in \mathbb{R}$, the set $\{x \in X \mid f(x) \le c\}$ is measurable.

(iii) For each $c \in \mathbb{R}$, the set $\{x \in X \mid f(x) > c\}$ is measurable.

(i) For each $c \in \mathbb{R}$, the set $\{x \in X \mid f(x) \ge c\}$ is measurable.

Each of these properties implies that for each extended real number c, the set $\{x \in X \mid f(x) = c\}$ is measurable.

Note. Since $\mathcal{M}$ is a σ-algebra, the proof of Proposition 18.1 is identical to the proof of Proposition 3.1.

Definition. Let $(X, \mathcal{M})$ be a measure space. An extended real-valued function f on X is said to be *measurable* (or *measurable with respect to* $\mathcal{M}$) provided one (and hence all) of the four statements of Proposition 18.1 holds.

Proposition 18.2. Let $(X, \mathcal{M})$ be a measure space and f a real-valued function on X. Then f is measurable if and only if for each open $\mathcal{O} \subset \mathbb{R}$, $f^{-1}(\mathcal{O})$ is measurable.

Idea of Proof. Every open set of real numbers is a countable disjoint union of open intervals. Any open interval $(a, b) \subset \mathbb{R}$ satisfies

$$f^{-1}((a,b)) = \{x \in X \mid f(x) < b\} \cap \{x \in X \mid f(x) > a\}.$$

Since $\mathcal{M}$ is a σ-algebra, the result follows. $\square$

Definition. For measure space $(X, \mathcal{M})$ and measurable subset $E \subset X$, extended real-valued function f defined on E is *measurable* if f is measurable on $(E, \mathcal{M}_E)$ where $\mathcal{M}_E$ is the collection of sets in $\mathcal{M}$ that are subsets of E.

Note. It easily follows that the restriction of measurable extended real-valued f on X to a measurable set $E \subset X$ is measurable on E and measurable on $X \setminus E$, and conversely.

Proposition 18.3. Let $(X, \mathcal{M}, \mu)$ be a complete measure space and X_0 a measurable subset of X for which $\mu(X \setminus X_0) = 0$. Then an extended real-valued function f on X is measurable if and only if its restriction to X_0 is measurable. In particular, if g and h are extended real-valued functions on X for which $g = h$ a.e. on X, then g is measurable if and only if h is measurable.

Note. The completeness of $(X, \mathcal{M}, \mu)$ is explicit in the proof of Proposition 18.3. In fact, Proposition 18.3 does not hold if the measure space is not complete (see Problem 18.2).

Theorem 18.4. Let $(X, \mathcal{M})$ be a measurable space and f and g measurable real-valued functions on X.

- Linearity: For any $\alpha, \beta \in \mathbb{R}$, $\alpha f + \beta g$ is measurable.
- Products: $f \cdot g$ is measurable.
- Maximum and Minimum: $\max\{f, g\}$ and $\min\{f, g\}$ are measurable.

Proof. The proof is the same as the result for Lebesgue measurable functions (Theorem 3.6). ∎

Note. Notice that Theorem 18.4 applies only to real-valued functions. In taking a linear combination $\alpha f + \beta g$, there is the concern of $\infty - \infty$ if we allow extended real-valued functions. Notice the blanket statement on page 361 that we bury the extended real values on sets of measure 0.

Proposition 18.5. Let $(X, \mathcal{M})$ be a measurable space, f a measurable real-valued function on X, and $\varphi : \mathbb{R} \to \mathbb{R}$ continuous. Then the composition $\varphi \circ f : X \to \mathbb{R}$ also is measurable.

Note. We will use Proposition 18.5 in Chapter 19 when we consider L^p spaces and use $\varphi(t) = |t|^p$, so that $\varphi \circ f = |f|^p$.

Theorem 18.6. Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{f_n\}$ a sequence of measurable functions on X for which $\{f_n\} \to f$ pointwise a.e. on X. If <u>either</u> the measure space $(X, \mathcal{M}, \mu)$ is complete <u>or</u> the convergence is pointwise on all of X, then f is measurable.

Note. Theorem 18.6 does not hold if the measure space is not complete (see Problem 18.3).

Corollary 18.7. Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{f_n\}$ be a sequence of measurable functions on X. Then the following functions are measurable:

$$\sup\{f_n\}, \inf\{f_n\}, \limsup\{f_n\}, \liminf\{f_n\}.$$

Note. The following two results are the parallels to results of the same name from Section 3.2.

The Simple Approximation Lemma.

Let $(X, \mathcal{M})$ be a measurable space and f a measurable function on X that is bounded on X. Then for each $\varepsilon > 0$, there are simple functions φ_ε and ψ_ε on X such that

$$\varphi_\varepsilon \leq f \leq \psi_\varepsilon \text{ and } 0 \leq \psi_\varepsilon - \varphi_\varepsilon < \varepsilon \text{ on } X.$$

The Simple Approximation Theorem.

Let $(X, \mathcal{M}, \mu)$ be a measure space and f a measurable function on X. Then there is a sequence $\{\psi_n\}$ of simple functions on X that converges pointwise on X to f and $|\psi_n| \leq |f|$ on X for all $n \in \mathbb{N}$.

(i) If X is σ-finite, then we may choose the sequence $\{\psi_n\}$ so that each ψ_ε vanishes outside a set of finite measure.

(ii) If f is nonnegative, we may choose the sequence $\{\psi_n\}$ to be increasing and each $\psi_n \geq 0$ on X.

Note. The proof of Egoroff's Theorem in the measure space setting is proven similarly to the proof in the Lebesgue measure setting—namely, using continuity and countable additivity of measure. Recall that it says that, on sets of finite measure, pointwise convergence is "nearly" uniform convergence.

Egoroff's Theorem.

Let $(X, \mathcal{M}, \mu)$ be a finite measure space and $\{f_n\}$ a sequence of measurable functions on X that converges pointwise a.e. on X to function f which is finite a.e. on X. Then for each $\varepsilon > 0$, there is a measurable subset X_ε of X for which $\{f_n\} \to f$ uniformly on X_ε and $\mu(X \setminus X_\varepsilon) < \varepsilon$.

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