Section 18.2. Integration of Nonnegative Measurable Functions

Note. We now define integrals of measurable functions on measure spaces. Though similar to the development of Lebesgue integration to Chapter 4, this section differs from the original approach in that we rely more heavily on simple functions. In a sense, we skip the step of considering bounded functions on sets of finite measure and define integrals of nonnegative measurable functions in terms of simple functions (which could have been done in the Lebesgue setting, as illustrated by Problem 4.24).

Definition. Let $(X, \mathcal{M}, \mu)$ be a measure space and ψ a nonnegative simple function on X. Define the *integral* of ψ over X, $\int_X \psi\, d\mu$ as follows: if $\psi = 0$ on X, define $\int_E \psi\, d\mu = 0$. Otherwise, let $c_1, c_2, \ldots, c_n$ be the positive distinct values taken by ψ on X and, for $1 \le k \le n$, define $E_k = \{x \in X \mid \psi(x) = c_k\}$. Define

$$\int_X \psi\, d\mu = \sum_{k=1}^n c_k \mu(E_k),$$

with the convention that the right-hand side is ∞ if, for some k, $\mu(E_k) = \infty$. For measurable $E \subset X$, the *integral* of ψ over E with respect to μ is

$$\int_X \psi \chi_E\, d\mu = \int_E \psi\, d\mu.$$

Note. By the use of the c_k's and the E_k's, we avoid a discussion of canonical form of a simple function.

Proposition 18.8. Let $(X, \mathcal{M}, \mu)$ be a measure space and let φ and ψ be nonnegative simple functions on X. If α and β are positive real numbers, then

$$\int_X (\alpha\psi + \beta\varphi)\, d\mu = \alpha \int_X \psi\, d\mu + \beta \int_X \varphi\, d\mu. \tag{2}$$

If A and B are disjoint measurable subsets of X, then

$$\int_{A \cup B} \psi\, d\mu = \int_A \psi\, d\mu + \int_B \psi\, d\mu. \tag{3}$$

In particular, if $X_0 \subset X$ is measurable and $\mu(X \setminus X_0) = 0$, then

$$\int_X \psi\, d\mu = \int_{X_0} \psi\, d\mu. \tag{4}$$

Furthermore, if $\psi \le \varphi$ a.e. on X, then

$$\int_X \psi\, d\mu \le \int_X \varphi\, d\mu. \tag{5}$$

Note. Now we extend the definition of integral from nonnegative simple functions to nonnegative extended real-valued measurable functions.

Definition. Let $(X, \mathcal{M}, \mu)$ be a measure space and f a nonnegative extended real-valued measurable function on X. The *integral of f over X with respect to μ*, denoted $\int_X f\, d\mu$, is the supremum of the integrals $\int_X \varphi\, d\mu$ as φ ranges over all simple functions φ for which $0 \le \varphi \le f$ on X. For measurable $E \subset X$, the *integral of f over E with respect to μ* is $\int_X f\chi_E\, d\mu$ and denoted $\int_E f\, d\mu$.

Lemma 18.2.A. Let $(X, \mathcal{M}, \mu)$ be a measure space, g and h nonnegative measurable functions on X, $X_0 \subset X$ measurable, and $\alpha > 0$ a real number. Then

$$\int_X \alpha g \, d\mu = \alpha \int_X g \, d\mu \tag{7}$$

$$\text{if } g \le h \text{ a.e. on } X \text{ then } \int_X g \, d\mu \le \int_X h \, d\mu \tag{8}$$

$$\int_X g \, d\mu = \int_{X_0} g \, d\mu \text{ if } \mu(X \setminus X_0) = 0. \tag{9}$$

Note. The proof of Lemma 18.2.A is left as an exercise (Exercise 4.17). Oddly enough, we leave the proof of linearity until after our proof of Fatou's Lemma.

Chebyshev's Inequality.

Let $(X, \mathcal{M}, \mu)$ be a measure space, f a nonnegative measurable function on X, and $\lambda > 0$ a real number. Then

$$\mu\{x \in X \mid f(x) > \lambda\} \le \frac{1}{\lambda} \int_X f \, d\mu.$$

Proposition 18.9. Let $(X, \mathcal{M})$ be a measure space and f nonnegative measurable function on X for which $\int_X f \, d\mu < \infty$. Then f is finite a.e. on X and $\{x \in X \mid f(x) > 0\}$ is σ-finite.

Note. We now state our new version of Fatou's Lemma, from which much of the rest of our theory of integration will follow.

Fatou's Lemma.

Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{f_n\}$ be a sequence of nonnegative measurable functions on X where $\{f_n\} \to f$ a.e. on X. Assume f is measurable. Then

$$\int_X f \, d\mu \leq \liminf \left(\int_X f_n \, d\mu \right).$$

Note. The assumption that f is measurable is necessary, unless Theorem 18.6 holds (in which case the measurability of f follows either from the pointwise convergence of $\{f_n\}$ on all of X or from the completeness of $(X, \mathcal{M}, \mu)$). Now for the proof.

Note. Now for our first convergence theorem in the abstract setting. We will have several such results similar to the setting of Lebesgue integration over $\mathbb{R}$. However, we will not have a result analogous to the Bounded Convergence Theorem.

The Monotone Convergence Theorem.

Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{f_n\}$ an increasing sequence (i.e., pointwise increasing) of nonnegative measurable functions on X. Define $f(x) = \lim_{n\to\infty} f_n(x)$ for each $x \in X$. Then

$$\lim_{n\to\infty} \left(\int_X f_n \, d\mu \right) = \int_X \left(\lim_{n\to\infty} f_n \right) \, d\mu = \int_X f \, d\mu.$$

Note. The proof of the Monotone Convergence Theorem is parallel to the proof given in Section 4.3 for the Lebesgue setting. It is not surprising that we now have the following (again, as in Section 4.3).

Beppo Levi's Lemma.

Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{f_n\}$ an increasing sequence of nonnegative measurable functions on X. If the sequence of integrals $\{\int_X f_n \, d\mu\}$ is bounded, then $\{f_n\}$ converges pointwise on X to a measurable function f that is finite a.e. on X and

$$\lim_{n\to\infty} \left(\int_X f_n \, d\mu \right) = \int_X \left(\lim_{n\to\infty} f_n \right) \, d\mu = \int_X f \, d\mu < \infty.$$

Note. The following is similar to Exercise 4.24 from Section 4.3 (but does not mention "finite support").

Proposition 18.10. Let $(X, \mathcal{M}, \mu)$ be a measure space and f a nonnegative measurable function on X. Then there is an increasing sequence $\{\psi_n\}$ of simple functions on X that converges pointwise on X to f and

$$\lim_{n\to\infty} \left(\int_X \psi_n \, d\mu \right) = \int_X f \, d\mu.$$

Note. If X is σ-finite (as $\mathbb{R}$ is under Lebesgue measure) then the simple φ_n by Proposition 18.10 may be chosen so that they vanish outside a set of finite measure (i.e., are of finite support) by part (i) of The Simple Approximation Theorem. This observation makes Proposition 18.10 an exact analogy with Exercise 4.24.

Note. So far, we have seen many properties of integrals in the abstract setting which parallel Lebesgue integration. Surprisingly, we have not yet shown linearity of integration for nonnegative measurable functions! The following result establishes linearity and uses some of the heavy equipment we have in place (namely, the Simple Approximation Theorem and the Monotone Convergence Theorem).

Proposition 18.11. Linearity of Integrals of Nonnegative Measurable Functions.

Let $(X, \mathcal{M}, \mu)$ be a measure space and f and g nonnegative measurable functions on X. If α and β are positive real numbers, then

$$\int_X (\alpha f + \beta g)\, d\mu = \alpha \int_X f\, d\mu + \beta \int_X g\, d\mu.$$

Definition. Let $(X, \mathcal{M}, \mu)$ be a measure space and f a nonnegative measurable function on X. Then f is *integrable* over X with respect to μ if $\int_X f\, d\mu < \infty$.

Note. It follows from Proposition 18.11 that the sum of nonnegative integrable functions is integrable. From Proposition 18.9, we see that if f is a nonnegative integrable function, then f is finite a.e. and vanishes outside a σ-finite set (i.e., f is of "σ-finite support").

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