Section 18.3. Integration of General Measurable Functions

Note. We now extend the results for nonnegative measurable functions to general measurable functions, as we did for Lebesgue integration in Section 4.4. As always, things will go as expected as long as we avoid "$\infty - \infty$."

Definition. Let $(X, \mathcal{M})$ be a measurable space and f a measurable function on X. The *positive part*, f^+, and *negative part*, f^-, of f are

$$f^+(x) = \max\{f(x), 0\} \text{ for } x \in X,$$

$$f^-(x) = \max\{-f(x), 0\} \text{ for } x \in X.$$

Note. Both f^+ and f^- are nonnegative measurable functions (and so can be dealt with using the techniques of the previous section). Also, $f = f^+ - f^-$ and $|f| = f^+ + f^-$. Since $0 \le f^+ \le |f|$ and $0 \le f^- \le |f|$, by (8) of "Lemma" we see that if $|f|$ is integrable then so are f^+ and f^-. Conversely, if f^+ and f^- are integrable then so is $|f|$ by Linearity of Integration of Nonnegative Measurable Functions (Proposition 18.11).

Definition. Let $(X, \mathcal{M}, \mu)$ be a measure space. A measurable function f on X is *integrable* over X with respect to μ if $|f|$ is integrable over X. For such a function we define the *integral* of f over X with respect to μ as

$$\int_X f\,d\mu = \int_X f^+\,d\mu - \int_X f^-\,d\mu.$$

For a measurable subset E of X, f is *integrable over* E if $f\chi_E$ is integrable over X and we define the *integral over* E of f as

$$\int_E f\,d\mu = \int_X f\chi_E\,d\mu.$$

The Integral Comparison Test.

Let $(X, \mathcal{M}, \mu)$ be a measure space and f a measurable function on X. If g is integrable over X and dominates f on X in the sense that $|f| \le g$ a.e. on X, then f is integrable over X and

$$\left|\int_X f\,d\mu\right| \le \int_X |f|\,d\mu \le \int_X g\,d\mu.$$

Note. If we wish to add measurable h and g on X (or take a linear combination of h and g), then we become concerned with "$\infty - \infty$." However, if h and g are integrable, then by Proposition 18.9, h and g are finite a.e. on X. Let $X_0 \subset X$ be the set on which both h and g are finite. Then $\mu(X \setminus X_0) = 0$. So by (9) from "Lemma" of the previous section,

$$\int_X (g+h)\,d\mu = \int_{X_0} (g+h)\,d\mu.$$

So, we can continue on our way with the study of *integrable* functions. Basically, as with Lebesgue integration and integrability, we have swept the nasty infinities under the measure zero $X \setminus X_0$ rug!

Theorem 18.12. Let $(X, \mathcal{M}, \mu)$ be a measure space and let f and g be integrable over X.

(Linearity) For $\alpha, \beta \in \mathbb{R}$, $\alpha f + \beta g$ is integrable over X and

$$\int_X (\alpha f + \beta g)\, d\mu = \alpha \int_X f\, d\mu + \beta \int_X g\, d\mu.$$

(Monotonicity) If $f \leq g$ a.e. on X, then

$$\int_X f\, d\mu \leq \int_X g\, d\mu.$$

(Additivity Over Domains) If A and B are disjoint measurable sets, then

$$\int_{A \cup B} f\, d\mu = \int_A f\, d\mu + \int_B f\, d\mu.$$

Note. It is easy to extend additivity using the Monotone Convergence Theorem.

Theorem 18.13. Countable Additivity over Domains of Integration. Let $(X, \mathcal{M}, \mu)$ be a measure space, let function f be integrable over X, and let $\{X_n\}_{n=1}^\infty$ be a disjoint countable collection of measurable sets whose union is X. Then

$$\int_X f\, d\mu = \int_{\cup X_k} f\, d\mu = \sum_{n=1}^\infty \left(\int_{X_k} f\, d\mu \right).$$

Theorem 18.14. Continuity of Integration.

Let $(X, \mathcal{M})$ be a measure space and let the function f be integrable over X.

(i) If $\{X_n\}_{n=1}^{\infty}$ is an ascending countable collection of measurable subsets of X whose union is X, then

$$\int_X f\,d\mu = \lim_{n\to\infty}\left(\int_{X_n} f\,d\mu\right).$$

(ii) If $\{X_n\}_{n=1}^{\infty}$ is a descending countable collection of measurable subsets of X, then

$$\int_{\cap X_n} f\,d\mu = \lim_{n\to\infty}\left(\int_{X_n} f\,d\mu\right).$$

Note. The only "concrete" examples of integrable functions we currently have are simple functions that vanish outside a set of finite measure. The following result gives a class of integrable functions similar to the class addressed in Section 4.2.

Theorem 18.15. Let $(X, \mathcal{M}, \mu)$ be a measure space and f a measurable function on X. If f is bounded on X and vanishes outside a set of finite measure, then f is integrable over X.

Note. We complete our study of integration of general measurable functions in the abstract setting with two convergence theorems (Lebesgue Dominated and Vitali).

The Lebesgue Dominated Convergence Theorem.

Let $(X, \mathcal{M}, \mu)$ be a measure space and let $\{f_n\}$ be a sequence of measurable functions on X for which $\{f_n\} \to f$ pointwise a.e. on X and suppose f is measurable. Assume there is a nonnegative function g that is integrable over X and dominates the sequence $\{f_n\}$ on X in the sense that $|f_n| \leq g$ a.e. on X for all $n \in \mathbb{N}$. Then f is integrable over X and

$$\lim_{n\to\infty} \left(\int_X f_n \, d\mu \right) = \int_X \left(\lim_{n\to\infty} f_n \right) d\mu = \int_X f \, d\mu.$$

Note. We now turn our attention to the Vitali Convergence Theorem. Recall that, in Sections 4.6 and 5.1, we had two versions of the Vitali Convergence Theorem. The first dealt with "uniform integrability" on a set of finite measure and the second dealt with uniform integrability and "tightness." We need new definitions and a preliminary result.

Definition. Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{f_n\}$ a sequence of functions on X, each of which is integrable over X. The sequence $\{f_n\}$ is *uniformly integrable* over X if for each $\varepsilon > 0$ there is a $\delta > 0$ such that for any $n \in \mathbb{N}$ and for any measurable subset E of X:

$$\text{if } \mu(E) < \delta \text{ then } \int_X |f_n| \, d\mu < \varepsilon.$$

The sequence $\{f_n\}$ is *tight* over X if for each $\varepsilon > 0$, there is a subset X_0 of X that has finite measure and for which

$$\int_{X \setminus X_0} |f_n| \, d\mu < \varepsilon \text{ for all } n \in \mathbb{N}.$$

Proposition 18.17. Let $(X, \mathcal{M}, \mu)$ be a measure space and let the function f be integrable over X. Then for each $\varepsilon > 0$, there is a $\delta > 0$ such that for any measurable subset E of X,

$$\text{if } \mu(E) < \delta \text{ then } \int_E |f| \, d\mu < \varepsilon. \tag{21}$$

Furthermore, for each $\varepsilon > 0$, there is a subset X_0 of X that has finite measure and

$$\int_{X \setminus X_0} |f| \, d\mu < \varepsilon.$$

The Vitali Convergence Theorem.

Let $(X, \mathcal{M}, \mu)$ be a measure space and let $\{f_n\}$ be a sequence of functions on X that is both uniformly integrable and tight over X. Suppose $\{f_n\} \to f$ pointwise a.e. on X and that f is integrable on X. Then

$$\lim_{n \to \infty} \left(\int_X f_n \, d\mu \right) = \int_X \left(\lim_{n \to \infty} f_n \right) d\mu = \int_X f \, d\mu.$$

Note. Here, the Vitali Convergence Theorem has a hypothesis that "f is integrable over X [and hence is measurable]." In the case of Lebesgue integration, the integrability of f followed from the a.e. pointwise convergence, uniform integrability, and tightness of $\{f_n\}$ (see page 98). Here is an example showing that this is not the case in general measure spaces.

Example. Let X be a set and E a proper nonempty subset of X. Define $\mathcal{M} = \{\varnothing, E, X \setminus E, X\}$. Then $\mathcal{M}$ is a σ-algebra. Define $\mu(\varnothing) = 0$, $\mu(E) = \mu(X \setminus E) = 1/2$, and $\mu(X) = 1$. Then μ is a measure. Define $f_n = n \cdot \chi_E - n \cdot \chi_{X\setminus E}$. Then $\{f_n\}$ is uniformly integrable and tight and converges pointwise to f where

$$f(x) = \begin{cases} \infty & \text{if } x \in E \\ -\infty & \text{if } x \in X \setminus E. \end{cases}$$

But then f is not integrable over E.

Note. There are added conditions in the measure space setting for which the integrability of f will follow. For example, consider:

Exercise 36. Let $\{f_n\}$ be a sequence of integrable functions on X that is uniformly integrable and tight. Suppose that $\{f_n\} \to f$ pointwise a.e. on X. *Suppose also that f is measurable and finite a.e. on X.* Then f is integrable over X.

Exercise 37. Let $\{f_n\}$ be a sequence of integrable functions on X that is uniformly integrable. Suppose that $\{f_n\} \to f$ pointwise a.e. on X. *Suppose also that f is measurable. Assume the measure space has the property that for each $\varepsilon > 0$, X is the union of a finite collection of measurable sets, each of measure at most ε.* Then f is integrable over X.

Corollary 18.18. Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{h_n\}$ a sequence of nonnegative integrable functions on X. Suppose that $\{h_n(x)\} \to 0$ for almost all $x \in X$. Then $\lim_{n\to\infty} \left(\int_X h_n \, d\mu \right) = 0$ if and only if $\{h_n\}$ is uniformly integrable and tight.

Note. The above result is the analogous to Corollary 5.2 from the Lebesgue integration setting.

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