Section 18.4. The Radon-Nikodym Theorem

Note. For $(X, \mathcal{M}, \mu)$ a measure space and f a nonnegative function on X that is measurable with respect to $\mathcal{M}$, the set function ν on $\mathcal{M}$ defined as $\nu(E) = \int_E f \, d\mu$ is a measure on $(X, \mathcal{M})$. This follows from the fact that $\nu(\varnothing) = \int_\varnothing f \, d\mu = 0$ and ν is countably additive by the linearity and the Monotone Convergence Theorem (see Exercise 18.45). The Radon-Nikodym Theorem says that, in a sense, any measure on $(X, \mathcal{M})$ results from the integration of a nonnegative function.

Definition. A measure ν on $(X, \mathcal{M})$ is *absolutely continuous* with respect to measure μ if for all $E \in \mathcal{M}$ with $\mu(E) = 0$, we have $\nu(E) = 0$. This is denoted $\nu \ll \mu$.

Proposition 18.19. Let $(X, \mathcal{M}, \mu)$ be a measure space and ν a finite measure on the measurable space $(X, \mathcal{M})$. Then ν is absolutely continuous with respect to μ if and only if for each $\varepsilon > 0$ there is a $\delta > 0$ such that for any set $E \in \mathcal{M}$, if $\mu(E) < \delta$ then $\nu(E) < \varepsilon$.

The Radon-Nikodym Theorem. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space and ν a σ-finite measure defined on the measurable space $(X, \mathcal{M})$ that is absolutely continuous with respect to μ. Then there is a nonnegative f on X that is measurable with respect to $\mathcal{M}$ for which

$$\nu(E) = \int_E f \, d\mu \text{ for all } E \in \mathcal{M}.$$

The function f is unique in the sense that if g is any nonnegative measurable function on X for which $\nu(E) = \int_E g \, d\mu$ for all $E \in \mathcal{M}$, then $g = f$ μ-a.e.

Definition. The function f of the Radon-Nikodym Theorem is the *Radon-Nikodym derivative* of ν with respect to μ, denoted $\dfrac{d\nu}{d\mu}$.

Note. The benefit of the Radon-Nikodym Theorem is that it allows us to express a measure in terms of an integral and we have an extensive theory of integrals. So we can use the properties of integrals to establish properties of the measure. As an application, we can use the Radon-Nikodym Theorem to introduce *complex valued measures* (see Exercise 18.4.B) and to address a conditional probability measure (even when conditioning on a probability 0 event; see my online notes on "The General Concept of Conditional Probability and Expectation" at: `http://faculty.etsu.edu/gardnerr/Probability/notes/Prob-5-3.pdf`).

Note. The assumption of σ-finiteness of μ and ν in the Radon-Nikodym Theorem is necessary, as the following example shows.

Example. Consider the measurable space $(X, \mathcal{M})$ where $X = [0, 1]$ and $\mathcal{M}$ is the collection of Lebesgue measurable subsets of $[0, 1]$. Define μ to be the counting measure on $\mathcal{M}$ (so for finite sets the measure is the cardinality and for infinite sets the measure is ∞). Then the only set of μ-measure 0 is $\varnothing$, and so every measure on $\mathcal{M}$ is absolutely continuous with respect to μ. Let m be Lebesgue measure on $\mathcal{M}$. Then by Exercise 18.60, there is no nonnegative Lebesgue measurable function f on X for which

$$m(E) = \int_E f\,d\mu \text{ for all } E \in \mathcal{M}.$$

Definition. Let ν be a signed measure on $(X, \mathcal{M})$ and let $\nu = \nu_1 - \nu_2$ be the Jordan decomposition of ν. Define the *absolute value of signed measure* ν as $|\nu| = \nu_1 + \nu_2$. If μ is a measure on $\mathcal{M}$, the *signed measure ν is absolutely continuous* with respect to μ if $|\nu|$ is absolutely continuous with respect to μ (or equivalently if both ν_1 and ν_2 are absolutely continuous with respect to μ).

Corollary 18.20. The Radon-Nikodym Theorem for Signed Measures. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space and ν a finite signed measure on measurable space $(X, \mathcal{M})$ that is absolutely continuous with respect to μ. Then there is a function f that is integrable over X with respect to μ and

$$\nu(E) = \int_E f\,d\mu \text{ for all } E \in \mathcal{M}.$$

Function f is unique up to a set of μ-measure zero.

Note. Recall from Section 17.2, that two measures μ and ν on measurable space $(X, \mathcal{M})$ are *mutually singular* (denoted $\mu \perp \nu$) if there are disjoint A and B in $\mathcal{M}$ for which $X = A \cup B$ and $\nu(A) = \mu(B) = 0$.

The Lebesgue Decomposition Theorem.

Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space and ν a σ-finite measure on the measurable space $(X, \mathcal{M})$. Then there is a measure ν_0 on $\mathcal{M}$ which is singular with respect to μ, and a measure ν_1 on $\mathcal{M}$ which is absolutely continuous with respect to μ, for which $\nu = \nu_0 + \nu_1$. The measures ν_0 and ν_1 are unique.

Note. The Radon-Nikodym Theorem is named for Johann Radon(December 16, 1887 to May 25, 1956) and Otto Nikodym (August 3, 1887 to May 4, 1974).

Johann Radon

Otto Nikodym

Radon first proved the result for the measure space $(\mathbb{R}^n, \mathcal{M}, \mu_n)$ in 1913 in his dissertation at the University of Vienna. He worked at the University of Hamburg, Erlanged, the University of Breslau. (See `http://www-groups.dcs.st-and.ac.uk/history/Biographies/Radon.html`.)

Nikodym proved the general case for a measure space in 1930 in "Sur une généralisation des intégrales de M. J. Rado" in *Fundamenta Mathematicae*, **15**, 131-179. Nikodym did his doctoral work at Warsaw University (Poland) and worked at the university of Kraków, Warsaw University, and at the private Kenyon College in

Ohio from 1948 to 1965. (See http://www-groups.dcs.st-and.ac.uk/history/Biographies/Nikodym.html.)

According to Wikipedia "The theorem is very important in extending the ideas of probability theory from probability masses and probability densities defined over real numbers to probability measures defined over arbitrary sets. It tells if and how it is possible to change from one probability measure to another. Specifically, the probability density function of a random variable is the RadonNikodym derivative of the induced measure with respect to some base measure (usually the Lebesgue measure for continuous random variables)." (https://en.wikipedia.org/wiki/Radon-Nikodym_theorem).

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