Section 18.4. The Radon-Nikodym Theorem (Partial)

Note. For \((X, \mathcal{M}, \mu)\) a measure space and \(f\) a nonnegative function on \(X\) that is measurable with respect to \(\mathcal{M}\), the set function \(\nu\) on \(\mathcal{M}\) defined as \(\nu(E) = \int_E f \, d\mu\) is a measure on \((X, \mathcal{M})\). This follows from the fact that \(\nu(\emptyset) = \int_\emptyset f \, d\mu = 0\) and \(\nu\) is countably additive by the Countable Additivity Over Domains of Integration (Theorem 18.13). The Radon-Nikodym Theorem says that, in a sense, any measure on \((X, \mathcal{M})\) results from the integration of a nonnegative function.

Definition. A measure \(\nu\) on \((X, \mathcal{M})\) is absolutely continuous with respect to measure \(\mu\) if for all \(E \in \mathcal{M}\) with \(\mu(E) = 0\), we have \(\nu(E) = 0\).

Proposition 18.19. Let \((X, \mathcal{M}, \mu)\) be a measure space and \(\nu\) a finite measure on the measurable space \((X, \mathcal{M})\). Then \(\nu\) is absolutely continuous with respect to \(\mu\) if and only if for each \(\epsilon > 0\) there is a \(\delta > 0\) such that for any set \(E \in \mathcal{M}\), if \(\mu(E) < \delta\) then \(\nu(E) < \epsilon\).
18.4. The Radon-Nikodym Theorem

The Radon-Nikodym Theorem. Let \((X, \mathcal{M}, \mu)\) be a \(\sigma\)-finite measure space and \(\nu\) a \(\sigma\)-finite measure defined on the measurable space \((X, \mathcal{M})\) that is absolutely continuous with respect to \(\mu\). Then there is a nonnegative \(f\) on \(X\) that is measurable with respect to \(\mathcal{M}\) for which

\[
\nu(E) = \int_E f \, d\mu \text{ for all } E \in \mathcal{M}.
\]

The function \(f\) is unique in the sense that if \(g\) is any nonnegative measurable function on \(X\) for which \(\nu(E) = \int_E g \, d\mu\) for all \(E \in \mathcal{M}\), then \(g = f \text{ } \mu\text{-a.e.}\)

Definition. The function \(f\) of the Radon-Nikodym Theorem is the Radon-Nikodym derivative of \(\nu\) with respect to \(\mu\), denoted \(\frac{d\nu}{d\mu}\).

Note. The assumption of \(\sigma\)-finiteness of \(\mu\) and \(\nu\) in the Radon-Nikodym Theorem is necessary, as the following example shows.

Example. Consider the measurable space \((X, \mathcal{M})\) where \(X = [0, 1]\) and \(\mathcal{M}\) is the collection of Lebesgue measurable subsets of \([0, 1]\). Define \(\mu\) to be the counting measure on \(\mathcal{M}\) (so for finite sets the measure is the cardinality and for infinite sets the measure is \(\infty\)). Then the only set of \(\mu\)-measure 0 is \(\emptyset\), and so every measure on \(\mathcal{M}\) is absolutely continuous with respect to \(\mu\). Let \(m\) be Lebesgue measure on \(\mathcal{M}\). Then by Exercise 18.60, there is no nonnegative Lebesgue measurable function \(f\) on \(X\) for which

\[
m(E) = \int_E f \, d\mu \text{ for all } E \in \mathcal{M}.
\]
Definition. Let $\nu$ be a signed measure on $(X, \mathcal{M})$ and let $\nu = \nu_1 - \nu_2$ be the Jordan decomposition of $\nu$. Define the absolute value of signed measure $\nu$ as $|\nu| = \nu_1 + \nu_2$. If $\mu$ is a measure on $\mathcal{M}$, the signed measure $\nu$ is absolutely continuous with respect to $\mu$ if $|\nu|$ is absolutely continuous with respect to $\mu$ (or equivalently if both $\nu_1$ and $\nu_2$ are absolutely continuous with respect to $\mu$).

Corollary 18.20. The Radon-Nikodym Theorem for Signed Measures.
Let $(X, \mathcal{M}, \mu)$ be a $\sigma$-finite measure space and $\nu$ a finite signed measure on measurable space $(X, \mathcal{M})$ that is absolutely continuous with respect to $\mu$. Then there is a function $f$ that is integrable over $X$ with respect to $\mu$ and

$$\nu(E) = \int_E f \, d\mu \text{ for all } E \in \mathcal{M}.$$ 

Note. Recall from Section 17.2, that two measures $\mu$ and $\nu$ on measurable space $(X, \mathcal{M})$ are mutually singular (denoted $\mu \perp \nu$) if there are disjoint $A$ and $B$ in $\mathcal{M}$ for which $X = A \cup B$ and $\nu(A) = \mu(B) = 0$.

The Lebesgue Decomposition Theorem.
Let $(X, \mathcal{M}, \mu)$ be a $\sigma$-finite measure space and $\nu$ a $\sigma$-finite measure on the measurable space $(X, \mathcal{M})$. Then there is a measure $\nu_0$ on $\mathcal{M}$ which is singular with respect to $\mu$, and a measure $\nu_1$ on $\mathcal{M}$ which is absolutely continuous with respect to $\mu$, for which $\nu = \nu_0 + \nu_1$. The measures $\nu_0$ and $\nu_1$ are unique.

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