Section 18.5. The Nikodym Metric Space: The Vitali-Hahn-Saks Theorem

Note. We construct a complete metric space from a finite measure space $(X, \mathcal{M}, \mu)$. We then use the metric to explore sequences of finite measures on $\mathcal{M}$. In spite of the similarity of the title of this section and the previous one, there is limited connection of the results of this section and the Radon-Nikodym Theorem.

Note. Recall the symmetric difference of sets A and B is $A \triangle B = (A \setminus B) \cup (B \setminus A)$ (see Section 2.4). We can easily show that $(A \triangle B) \triangle (B \triangle C) = A \triangle C$. So we can define a relation $\simeq$ on measure space $(X, \mathcal{M}, \mu)$ as $A \simeq B$ if $\mu(A \triangle B) \simeq 0$. This is clearly reflexive and symmetric and the above observation shows that it is transitive: $A \simeq B$ and $B \simeq C$ implies $A \simeq C$. So the relation is an equivalence relation on $\mathcal{M}$. Recall that an equivalence relation on a set partitions the set into equivalence classes. We (temporarily) denote the set of equivalence classes as $\mathcal{M}/_{\simeq}$ and denote the equivalence class containing A as $[A]$.

Note. Define the mapping $\rho_\mu : \mathcal{M}/_{\simeq} \times \mathcal{M}/_{\simeq} \to \mathbb{R}$ as $\rho_\mu([A], [B]) = \mu(A \triangle B)$.

Lemma 18.5.A. The map $\rho_\mu : \mathcal{M}/_{\simeq} \times \mathcal{M}/_{\simeq} \to \mathbb{R}$ is well-defined and is a metric on $\mathcal{M}/_{\simeq}$.

Definition. For finite measure space $(X, \mathcal{M}, \mu)$, the metric ρ_μ on $\mathcal{M}/_{\simeq}$ is the *Nikodym metric* and $(\mathcal{M}/_{\simeq}, \rho_\mu)$ is the *Nikodym metric space.*

Note. If ν is a finite measure on $\mathcal{M}$, where $(X, \mathcal{M}, \mu)$ is a finite measure space, and ν is absolutely continuous with respect to μ, then for $A, B \in \mathcal{M}$ with $A \simeq B$ we have $\mu(A \triangle B) = 0$ and s0 $\nu(A \triangle B) = 0$. So by additivity and monotonicity (Proposition 17.1)

$$\begin{aligned}
\nu(A) - \nu(B) &= \nu((A \cap B) \,\dot\cup\, (A \setminus B)) - \nu((A \cap B) \,\dot\cup\, (B \setminus A)) \\
&= \nu(A \cap B) + \nu(A \setminus B) - \nu(A \cap B) - \nu(B \setminus A) \\
&= \nu(A \setminus B) - \nu(B \setminus A) \\
&= 0 - 0 \text{ since } A \setminus B, B \setminus A \subset A \triangle B \text{ and } \nu(A \triangle B) = 0 \\
&= 0.
\end{aligned}$$

That is, if $A \simeq B$ then $\nu(A) = \nu(B)$. So we can properly define $\nu : \mathcal{M}/_{\simeq} \to \mathbb{R}$ as $\nu([A]) = \nu(A)$ for $A \in \mathcal{M}$.

Note. As we did in Chapter 7 when dealing with L^p spaces, we muddle the distinction between equivalence classes and elements of $\mathcal{M}$. So we simply represent $[A] \in \mathcal{M}/_{\simeq}$ as $A \in \mathcal{M}$ and hence treat ρ_μ as a metric on $\mathcal{M}$. We then use the notation $(\mathcal{M}, \rho_\mu)$ for the Nikodym metric space. We now show that this metric space is complete.

Theorem 18.21. Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Then the Nikodym metric space $(\mathcal{M}, \rho_\mu)$ is complete; that is, every Cauchy sequence converges.

Note. The next lemma says that if ν is a finite measure on $\mathcal{M}$, where $(X, \mathcal{M}, \mu)$ is a finite measure space, which is continuous at set $E_0 \in \mathcal{M}$ (treating ν as mapping metric space $(\mathcal{M}, \rho_\mu)$ into $\mathbb{R}$), then ν is uniformly continuous on $\mathcal{M}$.

Lemma 18.22. Let $(X, \mathcal{M}, \mu)$ be a finite measure space and ν a finite measure on $\mathcal{M}$. Let E_0 be a measurable set and $\varepsilon > 0$ and $\delta > 0$ be such that for any measurable set E,

$$\text{if } \rho_\mu(E, E_0) < \delta \text{ then } |\nu(E) - \nu(E_0)| < \varepsilon/4.$$

Then for any measurable sets A and B,

$$\text{if } \rho_\mu(A, B) < \delta \text{ then } |\nu(A) - \nu(B)| < \varepsilon.$$

Proposition 18.23. Let $(X, \mathcal{M}, \mu)$ be a finite measure space and ν a finite measure on $\mathcal{M}$ that is absolutely continuous with respect to μ. Then ν induces a properly defined (i.e., "well-defined"), uniformly continuous function on the Nikodym metric space associated with $(X, \mathcal{M}, \mu)$.

Note. For the remainder of this section we concentrate on sequences $\{\nu_n\}$ of finite measures on finite measure space $(X, \mathcal{M}, \nu)$.

Definition. Let $(X, \mathcal{M})$ be a measurable space. A sequence $\{\nu_n\}$ of measures on $\mathcal{M}$ *converges setwise on* $\mathcal{M}$ to set function ν if

$$\nu(E) = \lim_{n\to\infty} \nu_n(E) \text{ for all } E \in \mathcal{M}.$$

Definition. Let $(X, \mathcal{M}, \mu)$ be a finite measure space. A sequence $\{\nu_n\}$ of finite measures on $\mathcal{M}$, each of which is absolutely continuous with respect to μ, is *uniformly absolutely continuous* (or *equi absolutely continuous*) with respect to μ provided for each $\varepsilon > 0$ there is $\delta > 0$ such that for any $E \in \mathcal{M}$ and any $n \in \mathbb{N}$,

$$\text{if } \mu(E) < \delta, \text{ then } \nu_n(E) < \varepsilon.$$

Note. The next result relates uniformly absolutely continuous sequence of measures $\{\nu_n\}$ to an equicontinuous sequence $\{\nu_n : \mathcal{M} \to \mathbb{R}\}$ (with respect to ρ_μ) and to the sequence of Radon-Nikodym derivatives $\{d\mu/d\nu_n\}$. The proof is to be given in Exercise 18.64.

Proposition 18.24. Let $(X\, \mathcal{M}, \mu)$ be a finite measure space and $\{\nu_n\}$ a sequence of finite measures of $\mathcal{M}$ each of which is absolutely continuous with respect to μ. Then the following are equivalent:

(i) The sequence of measures $\{\nu_n\}$ is uniformly absolutely continuous with respect to μ.

(ii) The sequence of functions $\{\nu_n : \mathcal{M} \to \mathbb{R}\}$ is equicontinuous with respect to the Nikodym metric ρ_μ.

(iii) The sequence of Radon-Nikodym derivatives $\{d\mu/d\nu_n\}$ is uniformly integrable over X with respect to μ.

Note. The next result shows that if (uniformly absolutely continuous) sequence of finite measures $\{\nu_n\}$ converges setwise then the limit ν is a measure (and is uniformly absolutely continuous).

Theorem 18.25. Let $(X, \mathcal{M}, \mu)$ be a finite measure space and $\{\nu_n\}$ a sequence of finite measures on $\mathcal{M}$ that is uniformly absolutely continuous with respect to μ. If $\{\nu_n\}$ converges setwise on $\mathcal{M}$ to ν, then ν is a measure of $\mathcal{M}$ that is absolutely continuous with respect to μ.

Note. The next result shows that Theorem 18.25 holds if we drop the hypothesis that $\{v_n\}$ is uniformly absolutely continuous with respect to μ but require that $\{\nu_n(X)\}_{n=1}^{\infty}$ is a bounded sequence of real numbers. Royden and Fitzpatrick are smitten with the result and state (page 392): "The proof of this theorem rests beside the Uniform Boundedness Principle and the Open Mapping Theorem as one of the exceptional fruits of the Baire Category Theorem."

The Vitali-Hahn-Saks Theorem.

Let $(X, \mathcal{M}, \mu)$ be a finite measure space and $\{\nu_n\}$ a sequence of finite measures on $\mathcal{M}$, each if which is absolutely continuous with respect to μ. Suppose that $\{\nu_n(X)\}$ is bounded and $\{\nu_n\}$ converges setwise on $\mathcal{M}$ to ν. Then the sequence $\{\nu_n\}$ is uniformly continuous with respect to μ. Moreover, ν is a finite measure on $\mathcal{M}$ that is absolutely continuous with respect to μ.

Note. We now use the Vitali-Hahn-Saks Theorem to give a simplified result concerning the convergence of a sequence of measures.

Theorem 18.26. Nikodym.
Let $(X, \mathcal{M})$ be a measurable space and $\{\nu_n\}$ a sequence of finite measures on $\mathcal{M}$ which converges setwise on $\mathcal{M}$ to the set function ν. Assume $\{\nu_n(X)\}$ is bounded. Then ν is a measure on $\mathcal{M}$.

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