Chapter 19. General L^p Spaces: Completeness, Duality, and Weak Convergence

Note. As the title suggests, this chapter addresses L^p spaces of extended real valued functions defined on measure spaces $(X, \mathcal{M}, \mu)$. The "big results" have familiar names: Hölder's Inequality, Minkowski's Inequality, the Riesz-Fischer Theorem, the Riesz Representation Theorem. In addition, we finally have a Riesz Representation Theorem for L^∞ in Section 19.3. We see weak sequential convergence again and introduce the idea of weak sequential compactness (which gives us results similar to the familiar result from $\mathbb{R}$ that every bounded sequence of real numbers has a convergent subsequence).

Section 19.1. The Completeness of $L^p(X, \mu)$, $1 \leq p \leq \infty$

Note. We start with the same setup as we did with $L^p(E)$ where E is Lebesgue measurable. We give a version of the Riesz-Fisher Theorem for $L^p(X, \mu)$ where $1 \leq p \leq \infty$.

Definition. Let $(X, \mathcal{M}, \mu)$ be a measure space. Define $\mathcal{F}$ to be the set of all measurable extended real-valued functions on X that are finite a.e. on X. Define the relation $f \cong g$ if and only if $f = g$ a.e. on X.

Note. The relation $\cong$ is an equivalence relation, just as was the case in Chapter 7. So we denote the set of equivalence classes of $\mathcal{F}$ under $\cong$ as $\mathcal{F}/_{\cong}$. As usual, we can define linear combinations of equivalence classes, $\alpha[f] + \beta[g]$, where $\alpha, \beta \in \mathbb{R}$. To do so, we may have to ignore a set of measure 0 to avoid "$\infty - \infty$."

Definition. Define $L^p(X, \mu)$ as the set of equivalence classes $[f]$ for which $\int_E |f|^p \, d\mu < \infty$.

Note. We have defined $L^p(X, \mu)$ by using a *representative* f of $[f]$. However, if $f, g \in [f]$, then $g = g$ a.e. and $\int_E |f|^p \, d\mu = \int_E |g|^p \, d\mu$, so $L^p(X, \mu)$ is well-defined, and similarly $\|[f]\|_p$ is well-defined. However, we have not yet shown that $\|\cdot\|_p$ really is a norm.

Note. Since for all $a, b \in \mathbb{R}$ we have $|a + b|^p \leq 2^p(|f|^p + |g|^p)$ pointwise on X and so by the Integral Comparison Test (page 373)

$$\int_X |f + g|^p \, d\mu \leq 2^p \left(\int_X |f|^p \, d\mu + \int_X |g|^p \, d\mu \right) < \infty.$$

So $L^p(X, \mu)$ is closed under addition. It is also closed under multiplication by real numbers from Theorem 18.12. So $L^p(X, \mu)$ is a linear space. In addition, $\|[f]\|_p = 0$ if and only if $[f] = 0$ and $\|[\alpha f]\|_p = |\alpha| \|[f]\|_p$ for all $\alpha \in \mathbb{R}$.

Definition. Equivalence class $[f] \in \mathcal{F}$ is *essentially bounded* if there is some $M \geq 0$ for which $|f| \leq M$ a.e. on X. Such M is called an *essential upper bound* for $[f]$. Define $L^\infty(X, u)$ as the set of all equivalence classes $[f]$ for which f is essentially bounded. For $[f] \in L^\infty(X,\mu)$, define the *norm*

$$\|f\|_\infty = \inf\{M \mid M \geq 0, |f| \leq M \text{ a.e. on } X\}.$$

Note. As above, $L^\infty(X,\mu)$ and $\|[f]\|_\infty$ are well-defined. In addition, the properties of a norm are easily seen to be satisfied by $\|\cdot\|_\infty$, so this really is a norm.

Note. We now drop the equivalence class verbiage and simply talk about functions f as elements of $L^p(X,\mu)$. The proof of the following is similar to the proofs given in Section 7.2, "The Inequalities of Young, Hölder, and Minkowski," for the corresponding results in the setting of Lebesgue measure.

Theorem 19.1. Let $(X,\mathcal{M},\mu)$ be a measure space, $1 \leq p \leq \infty$, and q the conjugate of p (that is, $\frac{1}{p} + \frac{1}{q} = 1$). If $f \in L^p(X,\mu)$ and $g \in L^q(X,\mu)$, then the product $fg \in L^1(X,\mu)$ and:

- **(i)** Hölder's Inequality. $\int_X |fg|\, d\mu = \|fg\|_1 \leq \|f\|_p\|g\|_q$. Moreover, if $f \neq 0$, the function $f^* = \|f\|_p^{1-p}\text{sgn}(f)|F|^{p-1} \in L^q(X,\mu)$, $\int_X ff^*\, d\mu = \|f\|_p$ and $\|f^*\|_q = 1$.

- **(ii)** Minkowski's Inequality. For $1 \leq p \leq \infty$ and $f, g \in L^p(X,\mu)$, $\|f+g\|_p \leq \|f\|_p + \|g\|_p$. Therefore $L^p(X,\mu)$ is a normed linear space.

- **(iii)** The Cauchy-Schwarz Inequality. Let f and g be measurable functions on X for which f^2 and g^2 are integrable over X. Then their product fg also is integrable over X and $\int_E |fg|\, d\mu \leq \sqrt{\int_X f^2\, d\mu}\sqrt{\int_X g^2\, d\mu}$.

Note. The following is a generalization of Corollary 7.3 (page 142) and its proof is identical to that of Corollary 7.3.

Corollary 19.2. Let $(X, \mathcal{M}, \mu)$ be a finite measure space and $1 \le p_1 < p_2 \le \infty$. Then $L^{p_2}(X, \mu) \subset L^{p_1}(X, \mu)$. Moreover, for

$$c = [\mu(X)]^{(p_2 - p_1)/(p_1 p_2)} \text{ if } p_2 < \infty \text{ and } c = [\mu(X)]^{1/p_2} \text{ if } p_2 = \infty$$

we have that $\|f\|_{p_1} \le \|f\|_{p_2}$ for all $f \in L^{p_2}(X, \mu)$.

Note. Recall the following from Section 4.6, "Uniform Integrability: The Vitali Convergence Theorem."

Definition. A family $\mathcal{F}$ of measurable functions on measure space $(X, \mathcal{M}, \mu)$ is *uniformly integrable* over measurable $E \subset X$ if for each $\varepsilon > 0$ there is a $\delta > 0$ such that for all $f \in \mathcal{F}$ we have

$$\text{if } A \subset E \text{ is measurable and } \mu(A) < \delta \text{ then } \int_A |f|\, d\mu < \varepsilon.$$

Corollary 19.3. Let $(X, \mathcal{M}, \mu)$ be a measure space and $1 < p \le \infty$. If $\{f_n\}$ is a bounded sequence of functions in $L^p(X, \mu)$, then $\{f_n\}$ is uniformly integrable over X.

Note. We see from the proof of Corollary 19.3 that it in fact holds for any family $\mathcal{F}$ of measurable functions in $L^p(X, \mu)$ as long as the family of functions is bounded.

Definition. For a linear space V with norm $\|\cdot\|$, a sequence $\{v_k\} \subset V$ is *rapidly Cauchy* if there is a convergent series of positive real numbers $\sum_{k=1}^{\infty} \varepsilon_k < \infty$ for which $\|v_{k+1} - v_k\| \leq \varepsilon_k^2$ for all $k \in \mathbb{N}$.

Note. We showed that $L^p(E)$ for Lebesgue measurable E is complete (The Riesz-Fischer Theorem) using rapidly Cauchy sequences. We follow the same approach here. Recall that, in a normed linear space, every rapidly Cauchy sequence is Cauchy, and every Cauchy sequence has a rapidly Cauchy subsequence. This is Proposition 7.5.

Lemma 19.4. Let $(X, \mathcal{M}, \mu)$ be a measure space and $1 \leq p \leq \infty$. Then every rapidly Cauchy sequence in $L^p(X, \mu)$ converges to a function in $L^p(X, \mu)$, both with respect to the $L^p(X, \mu)$ norm and pointwise a.e. in X.

The Reisz-Fischer Theorem.

Let $(X, \mathcal{M}, \mu)$ be a measure space and $1 \leq p \leq \infty$. Then $L^p(X, \mu)$ is a Banach space. Moreover, if a sequence in $L^p(X, \mu)$ converges in $L^p(X, \mu)$ to $f \in L^p(X, \mu)$, then a subsequence converges pointwise a.e. on X to f.

Note. The following uses the Simple Approximation Theorem and shows the central role that simple functions play.

Theorem 19.5. Let $(X,\mathcal{M},\mu)$ be a measure space and $1 \le p < \infty$. Then the subspace of simple functions on X that vanish outside a set of finite measure is dense in $L^p(X,\mu)$.

Recall. A sequence $\{f_n\}$ is *tight* over X in $(X,\mathcal{M},\mu)$ if for each $\varepsilon > 0$, there is a subset X_0 of X that has finite measure and $\displaystyle\int_{X\setminus X_0} |f_n|\,d\mu < \varepsilon$ for all $n \in \mathbb{N}$.

Note. The proof of the following uses the Vitali Convergence Theorem and relates convergence of a sequence with respect to the L^p norm to uniform integrability and tightness. We leave the proof as Exercise 19.1.B.

The Vitali L^p Convergence Theorem.
Let $(X,\mathcal{M},\mu)$ be a measure space and $1 \le p < \infty$. Suppose $\{f_n\}$ is a sequence in $L^p(X,\mu)$ that converges pointwise a.e. to f and suppose $f \in L^p(X,\mu)$. Then $\{f_n\} \to f$ in $L^p(X,\mu)$ if and only if $\{|f|^p\}$ is uniformly continuous and tight.

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