Section 19.2. The Riesz Representation Theorem for the Dual of $L^p(X, \mu)$, $1 \le p < \infty$

Note. We now carry the result of Section 8.1, "The Riesz Representation for the Dual of L^p, $1 \le p < \infty$," over to the setting of $L^p(X, \mu)$. The result is the same as in Section 8.1, but with the added hypothesis that $(X, \mathcal{M}, \mu)$ is σ-finite (of course, $(\mathbb{R}, \mathcal{M}, \mu)$ is σ-finite where m is Lebesgue measure). Recall that we denote the set of all bounded linear functionals on $L^p(X, \mu)$ as $(L^p(X, \mu))^*$ (the *dual* of $L^p(X, \mu)$).

Definition. For $1 \le p < \infty$, let $f \in L^q(X, \mu)$ where q is conjugate to p. Define the *linear functional* $T_f : L^p(X, \mu) \to \mathbb{R}$ by

$$T_f(g) = \int_X fg \, d\mu \text{ for all } g \in L^p(X, \mu).$$

Note. By Hölder's Inequality (Theorem 19.1(i)) T_f is a bounded linear functional on $L^p(X, \mu)$ and its norm is bounded by $\|f\|_q$. By the "moreover" part of the Hölder's Inequality, with $g = f^* \in L^p(X, \mu)$ we have $T_f(f^*) = \int_X f f^* \, d\mu = \|f\|_p$ and so $\|T_f\| = \|f\|_p$. So $T : L^q(X, u) \to (L^p(X, \mu))^*$ defined by $T(f) = T_f$ is an isometry. We now show that T is, in fact, an onto isometry provided $(X, \mathcal{M}, \mu)$ is σ-fine (that is, T is an isometric isometry in this case).

The Riesz Representation Theorem for the Dual of $L^p(X, \mu)$.
Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space, let $1 \le p < \infty$, and let q be the conjugate of p. For $f \in L^q(X, \mu)$ define $T_f \in (L^p(X, \mu))^*$ as $T_f(g) = \int_X fg\, d\mu$. Then $T : L^q(X, \mu) \to (L^p(X, \mu))^*$, defined as $T(f) = T_f$, is an isometric isomorphism.

Note. We give the proof of this general version of the Riesz Representation Theorem below. In Section 8.1 we first proved the Riesz Representation Theorem for set $[a, b]$ in Theorem 8.5. In the current setting we have a σ-finite measure space and will use the Radon-Nikodym Theorem (which requires a σ-finite measure space) to express a bounded linear functional as an integral. We need one lemma before presenting the lengthy proof of the Riesz Representation Theorem.

Lemma 19.6. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space and let $1 \le p < \infty$. For f an integrable function over X, suppose there is $M \le 0$ such that for every simple function g on X that vanishes outside a set of finite measure, we have $\left|\int_X fg\, d\mu\right| \le M\|g\|_p$. Then $f \in L^q(X, \mu)$ where q is the conjugate of p. Moreover, $\|f\|_q \le M$.

Note. We are new ready to give a proof of the Reisz Representation Theorem.

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