Section 19.3. The Kantorovitch Representation Theorem for the Dual of $L^\infty(X, \mu)$

Note. In Section 8.1, "The Riesz Representation for the Dual of L^p, $1 \leq p < \infty$," we proved that the dual space of $L^p(E)$ is $L^q(E)$ for $1 \leq p < \infty$ and $1/p + 1/q = 1$ where $E \subset \mathbb{R}$ is a Lebesgue measurable set. In Section 19.2, "The Riesz Representation for the Dual of $L^p(X, \mu)$, $1 \leq p < \infty$," we proved that the dual space of $L^p(X, \mu)$ is $L^q(X, \mu)$ for $1 \leq p < \infty$. For $1 < p < \infty$, we have L^p and L^q as duals of each other (well, up to isometric isomorphism). The dual of L^1 is L^∞, but we will see that the dual of L^∞ is *not* L^1. In this section we characterize the dual of L^∞. Surprisingly, it involves signed measures.

Definition. Let $(X, \mathcal{M})$ be a measurable space and the set function $\nu : \mathcal{M} \to \mathbb{R}$ be finitely additive. For $E \in \mathcal{M}$, the *total variation* of ν over E, denoted $|\nu|(E)$, is

$$|\nu|(E) = \sup \sum_{k=1}^{n} |\nu((E_k)|,$$

where the supremum is taken over finite disjoint collections $\{E_k\}_{k=1}^{\infty}$ of sets in $\mathcal{M}$ that are contained in E. We say ν is a *bounded finitely additive signed measure* provided $|\nu|(X) < \infty$. The total variation of such ν is denoted $\|\nu\|_{\text{var}}$ and also defined as $\|\nu\|_{\text{var}} = |\nu|(E)$.

Note. Let $\nu : \mathcal{M} \to \mathbb{R}$ be a measure. Then for finite disjoint collections $\{E_k\}_{k=1}^n$ of sets in $\mathcal{M}$ that are contained in $E \in \mathcal{M}$ we have

$$\begin{aligned}\nu\left(\cup_{k=1}^n E_k\right) &= \sum_{k=1}^n \nu(E_k) \text{ by countable additivity} \\ &\leq \nu(E) \text{ by monotonicity, Proposition 17.1(ii),}\end{aligned}$$

so $\sup \sum_{k=1}^n |\nu(E_k)| = \sum_{k=1}^n \nu(E_k) \leq \nu(E)$. With $\{E_1\} = \{E\}$, $\sum_{k=1} \nu(E_k) = \nu(E_1) = \mu(E)$, and hence $|\nu|(E) = \nu(E)$ for all $E \in \mathcal{M}$. In particular, $\|\nu\|_{\text{var}} = |\nu|(X) = \nu(X)$.

Definition. If $\nu : \mathcal{M} \to \mathbb{R}$ is a signed measure with Jordan decomposition $\nu = \nu^+ - \nu^-$. We define the total variation of ν in Section 17.2, "Signed Measures: The Hahn and Jordan Decompositions," as $\|\nu\|_{\text{var}} = |\nu|(X) = \nu^+(X) + \nu^-(X)$ (see also Exercise 17.16), consistent with the definition here.

Note. We now, for the first time, define integrals with respect to signed measures. Recall that, by definition (see Section 17.2), a signed measure assumes at most one of the values $+\infty, -\infty$.

Definition. Let $\nu : \mathcal{M} \to \mathbb{R}$ be a bounded finite additive signed measure on $\mathcal{M}$ and let $\varphi = \sum_{k=1}^n c_k \chi_{E_k}$ be a measurable simple function. Define the *integral* of φ over X with respect to ν as

$$\int_X \varphi \, d\nu = \sum_{k=1}^n c_k \nu(E_k).$$

Note. The previous definition does not appeal to a canonical representation of φ (as we did directly in Section 4.1 and indirectly in Section 18.2), so we need to address whether $\int_X \varphi\,d\nu$ is well-defined (or "properly defined" as Royden and Fitzpatrick say). It is, in fact, well-defined. This was shown in Lemma 4.1 for the Lebesgue integral of simple functions. The proof of Lemma 4.1 only uses finite additivity of Lebesgue measure (and some elementary set theory), and we have finite additivity (in fact, countable additivity) by the definition of a signed measure. So Lemma 4.1 and the fact that $\int_X \varphi\,d\nu$ is well-defined holds here. Linearity and monotonicity of simple functions in the Lebesgue setting is given in Proposition 4.2 and the proof only depends on Lemma 4.1. So we can establish linearity and monotonicity for signed measures in an identical way.

Note. For $\varphi = \sum_{k=1}^n c_k \chi_{E_k}$ we have

$$\begin{aligned}\left|\int_X \varphi\,d\nu\right| &= \left|\sum_{k=1}^n c_k\nu(E_k)\right| \le \sum_{k=1}^n |c_k||\nu(E_k)| \le \|\varphi\|_\infty \sum_{k=1}^n |\nu(E_k)| \\ &\le \|\varphi\|_\infty \left(\sum_{k=1}^n |\nu(E_k) + |\nu\left(X \setminus \cup_{k=1}^n E_k\right)|\right) \\ &\le \|\varphi\|_\infty \sup \sum_{k=1}^m |\nu(E_k)| = \|\varphi\|_\infty \|\nu\|_{\text{var}}. \qquad (17)\end{aligned}$$

Note. Let $f : X \to \mathbb{R}$ be a bounded measurable function. By the Simple Approximation Lemma (Section 18.1) there are sequences $\{\psi_n\}$ and $\{\phi_n\}$ of simple functions on X for which $\varphi_n \le \varphi_{n+1} \le f \le \psi_{n+1} \le \psi_n$ and $0 \le \psi_n - \varphi_n \le 1/n$ on X for all $n \in \mathbb{N}$. So there is a sequence, say $\{\varphi_n\}$, which converges uniformly to f

on X. By the previous note and linearity we have for all $n, k \in \mathbb{N}$:

$$\left| \int_X (\varphi_{n+k} - \varphi_n)\, d\nu \right| = \left| \int_X \varphi_{n+k}\, d\nu - \int_X \varphi_n\, d\nu \right| \le \|\nu\|_{\text{var}} \|\varphi_{n+k} - \varphi_n\|_\infty.$$

Definition. Let $f : X \to \mathbb{R}$ be a bounded measurable function on X. Let ν be a bounded finitely additive signed measure on $(X, \mathcal{M})$. Define the *integral*

$$\int_X f\, d\nu = \lim_{n\to\infty} \int_X \varphi_n\, d\nu$$

where $\{\varphi_n\}$ is a sequence of simple functions which converges uniformly to f on X.

Note. Since ν is bounded then $\|\nu\|_{\text{var}} < \infty$. Since $\{\varphi_n\}$ converges uniformly on X to f then it converges to f with respect to $\|\cdot\|_\infty$ on X. So if $\{\psi_n\}$ is also a sequence converging uniformly to f then for given $\varepsilon > 0$ there are $N_1, N_2 \in \mathbb{N}$ such that for all $n \ge N_1$ we have $\|\varphi_n - f\|_\infty < \varepsilon/2$ and for all $n \ge N_2$ we have $\|\psi_n - f\|_\infty < \varepsilon/2$. So for $n \ge \max\{N_1, N_2\}$ we have

$$\|\varphi_n - \psi_n\|_\infty = \|\varphi_n - f + f - \psi_n\|_\infty \le \|\varphi_n - f\|_\infty + \|\psi_n - f\|_\infty < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

and

$$\begin{aligned} \left| \int_X \varphi_n\, d\nu - \int_X \psi_n\, d\nu \right| &= \left| \int_X (\varphi_n - \psi_n)\, d\nu \right| \\ &\le \|\nu\|_{\text{var}} \|\varphi_n - \psi_n\|_\infty \text{ by (17)} \\ &< \|\nu\|_{\text{var}}\, \varepsilon. \end{aligned}$$

Since $\|\nu\|_{\text{var}} < \infty$ (because ν is a bounded finitely additive signed measure), then $\lim_{n\to\infty} \int_X \varphi_n\, d\nu = \lim_{n\to\infty} \int_X \psi_n\, d\nu$ and so $\int_X f\, d\nu$ is well-defined (that is, independent of the sequence of simple functions which converges uniformly to f on X).

Note. As in the Riesz Representation Theorem of Section 19.2, we expect to find an isometric isomorphism T from $L^\infty(X,\mu)$ to its dual space where T is based on bounded linear functionals of the form T_f where T_f is an integral and f is an element of the dual space. We would also expect the integrals to be with respect to measure μ but, we we will see, the integrals in fact involve bounded finitely addition signed measures. Since $L^\infty(X,\mu)$ involves equivalence classes of essentially bounded measurable functions on X, we will need $\int_X f\,d\nu = \int_X g\,d\nu$ if and only if $f = g$ μ-a.e. on X (so that a bounded linear functional defined using an integral with respect to ν will map functions in the same equivalence class of $L^\infty(X,\mu)$ to the same real number; that is, so the bounded linear functionals are well-defined). But if there is $E \in \mathcal{M}$ for which $\mu(E) = 0$ and $\nu(E) \neq 0$ then we can take $f = 0$ on X and $g = \begin{cases} 1 & \text{on } E \\ 0 & \text{on } X \setminus E \end{cases}$ so that $f = g$ μ-a.e. but

$$\int_X f\,d\nu = 0 \neq \nu(E) = \int_E a\,d\nu = \int_E g\,d\nu = \int_X g\chi_E\,d\nu = \int_X f\,d\nu.$$

In order to avoid this, we only consider bounded finitely additive signed measures which avoid this behavior on sets of μ-measure zero. Recall that ν is *absolutely continuous* with respect to μ if $E \in \mathcal{M}$ and $\mu(E) = 0$ implies $\nu(E) = 0$ (see Section 18.4, "The Radon-Nikodym Theorem").

Definition. Let $(X,\mathcal{M},\mu)$ be a measure space. Denote by $\mathcal{BFA}(X,\mathcal{M},\mu)$ the *normed linear space of bounded finitely additive signed measures* ν on $\mathcal{M}$ that are absolutely continuous with respect to μ in the sense that if $E \in \mathcal{M}$ and $\mu(E) = 0$, then $\nu(E) = 0$. The norm of $\nu \in \mathcal{BFA}(X,\mathcal{M},\mu)$ is the total variation norm $\|\nu\|_{\text{var}}$.

Note. We leave as Exercise 19.3.A the proof that $\mathcal{BFA}(X,\mathcal{M},\mu)$ is in fact a normed linear space. Notice that if φ and ψ are simple functions on X which are equal to μ-a.e. then they are equal ν-a.e. and so $\int_X \varphi\,d\nu = \int_X \psi\,d\nu$. Therefore is f and g are bounded (or essentially bounded) on X and equal μ-a.e. then (since the integral is defined in terms of simple functions) $\int_X f\,d\nu = \int_X g\,d\nu$. Therefore, the integral of an element of $L^\infty(X,\mu)$ (that is, a equivalence class of μ-a.e. equal essentially bounded functions) is well-defined. Also, from (17), for $f \in L^\infty(X,\mu)$ and $\nu \in \mathcal{BFA}(X,\mathcal{M},\mu)$ we have $\left|\int_X f\,d\nu\right| \le \|\nu\|_{\text{var}}\|f\|_\infty$. We now characterize the duals space of $L^\infty(X,\mu)$ similar to the Reisz Representation Theorem for $L^p(X,\mu)$ where $1 \le p < \infty$ as stated in the previous section

Theorem 19.7. The Kantorovitch Representation Theorem.
let $(X,\mathcal{M},\mu)$ be a measure space. For signed measure $\nu \in \mathcal{BFA}(X,\mathcal{M},\mu)$ define $T_\nu : L^\infty(X,\mu) \to \mathbb{R}$ by

$$T_\nu(f) = \int_X f\,d\nu \text{ for all } r \in L^\infty(X,\mu).$$

Then $T : \mathcal{BFA}(X,\mathcal{M},\mu) \to L^\infty(X,\mu)^*$, which maps ν to T_ν, is an isometric isomorphism of the normed linear space $\mathcal{BFA}(X,\mathcal{M},\mu)$ onto the dual of $L^\infty(X,\mu)$.

Note. So the dual space of $L^\infty(X,\mu)$ "is" the linear space $\mathcal{BFA}(A,\mathcal{M},\mu)$ of all bounded finitely additive measures on $\mathcal{M}$ that are absolutely continuous with respect to μ.

Note. On page 496, Royden and Fitzpatrick give an argument based on the Kantorovitch Representation Theorem and a result of Chapter 14, "Duality for Normed Linear Spaces," that there is a bounded set function on the Lebesgue measurable subsets of $[a,b]$ that is absolutely continuous with respect to Lebesgue measure, is finitely additive by not countably additive. Though the existence of such a set function is guaranteed, no such function has been explicitly exhibited.

Note. Leonid V. Kantorovitch (January 16, 1912–April 7, 1986; also spelled "Kantorovich") was a Russian mathematician active in research from the age of 15. He contributed to mathematics, economics, and computer science and published over 300 papers and books. He made contributions in functional analysis, approximation theory, and numerical analysis/linear programming. His developed an interest in economics in 1938 and ultimately was a joint winner of the Nobel Prize in economics in 1975. The result of this section was apparently in:

> L. V. Kantorovich and B. S. Vulich, "Sur la représentation des operations linéaires," *Compositio Math.*, **5** (1938), 119–165.

For a reference to this, see page 572 of G. G. Lorentz and D. G. Wertheim, "Representation of Linear Functions on Köthe Spaces, *Canadian Journal of Mathematics* **5**(4) (1953) 568–575, available on `books.google.com` (as of January 20, 2019).

This information is based on the MacTutor History of Mathematics Archive www-history.mcs.st-and.ac.uk/Biographies/Kantorovich.html (where the image was accessed) and en.wikipedia.org/wiki/Leonid_Kantorovich.

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