Section 19.4. Weak Sequential Compactness in $L^p(X, \mu)$, $1 < p < \infty$

Note. We prove two results in the brief section. First we consider reflexive Banach spaces, a topic originally defined in Section 14.1, "Linear Functionals, Bounded Linear Functionals, and Weak Topologies," and explored in Section 14.3, "Reflexive Banach Spaces and Weak Sequential Convergence." Second, we generalize Theorem 8.15 from $L^p(E, m)$ where $1 < p < \infty$ and $E \subset \mathbb{R}$ to $L^p(X, \mu)$ where $1 < p < \infty$ and $(X, \mathcal{M}, \mu)$ is σ-finite. We start by recalling some definitions.

Note. Recall that the *dual space* of a normed linear space X is the normed linear space of all bounded linear functionals on X. The dual space of X is denoted X^* and the deal space of X^* is denoted X^{**}. The linear operator $J : X \to X^{**}$ defined by

$$J(x)(\psi) = \psi(x) \text{ for all } x \in X, \psi \in X^*$$

is the *natural embedding* of X into X^{**}. The space X is *reflexive* if $J(X) = X^{**}$ (see Section 14.1).

Note. Given our experience with L^p spaces in the setting of Lebesgue measure, we would expect $L^p(X, \mu)$ to be reflexive for $1 < p < \infty$ (remember, the dual space of L^∞ is *not* L^1, as shown in the previous section). In fact, this is the case with one added hypothesis on measure space $(X, \mathcal{M}, \mu)$, as follows.

Theorem 19.8. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space and $1 < p < \infty$. Then $L^p(X,\mu)$ is a reflexive Banach space.

Note. Theorem 14.17 states that in a reflexive Banach space, every bounded sequence in X has a weakly convergent subsequence (sequence $\{x_n\}$ is *weakly convergent* to $x \in X$ if $\lim_{n\to\infty} \psi(x_n) = \psi(x)$ for all $\psi \in X^*$). Given Theorem 19.8, we easily have the following.

The Riesz Weak Compactness Theorem.
Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space and $1 < p < \infty$. Then every bounded sequence in $L^p(X,\mu)$ has a weakly convergent subsequence; that is. if $\{f_n\}$ is a bounded sequence in $L^p(X,\mu)$, then there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ and a function $f \in L^p(X,\mu)$ for which

$$\lim_{k\to\infty} \int_X f_{n_k} g \, d\mu = \int_X fg \, d\mu \text{ for all } g \in L^q(X,\mu),$$

where $1/p + 1/q = 1$.

Note. In fact, we can extend some other results from $L^p(\mathbb{R}, \mathcal{M}, n)$ to $L^p(X, \mathcal{M}, \mu)$, provided $(X, \mathcal{M}, \mu)$ is σ-finite. The following three results follow from identical proofs from the $L^p(\mathbb{R}, \mathcal{M}, m)$ setting (the second result corresponds to Corollary 8.13).

The Radon-Riesz Theorem. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space, $1 < p < \infty$, and $\{f_n\}$ a sequence in $L^p(X,\mu)$ that converges weakly in $L^p(X,\mu)$ to f. Then

$$\{f_n\} \text{ converges strongly in } L^p(X,\mu) \text{ to } f$$

$$\text{if and only if}$$

$$\lim_{n\to\infty} \|f_n\|_p = \|f\|_p.$$

Corollary 19.9. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space, $1 < p < \infty$ and $\{f_n\}$ a sequence in $L^p(X,\mu)$ that converges weakly in $L^p(X,\mu)$ to f. Then a subsequence of $\{f_n\}$ converges strongly in $L^p(X,\mu)$ to f if and only if $\|f\|_p = \liminf \|f_n\|_p$.

The Banach-Saks Theorem. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space, $1 < p < \infty$, and $\{f_n\}$ a sequence in $L^p(X,\mu)$ that converges weakly in $L^p(X,\mu)$ to f. Then there is a subsequence $\{f_{n_k}\}$ for which the sequence of Cesàro means converges strongly in $L^p(X,\mu)$ to f, that is

$$\lim_{k\to\infty} \frac{f_{n_1} + f_{n_2} + \cdots + f_{n_k}}{k} = f \text{ strongly in } L^p(X,\mu).$$

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