Section 19.5. Weak Sequential Compactness in $L^1(X, \mu)$: The Dunford-Pettis Theorem

Note. The Riesz Weak Compactness Theorem of the previous section implies that for σ-finite $(X, \mathcal{M}, \mu)$, every bounded sequence in $L^p(X, \mu)$, where $1 < p < \infty$, has a weakly convergent subsequence. In this section, we give a sufficient condition for a bounded sequence in $L^1(X, \mu)$ to have a weakly convergent subsequence (by the Eberlein-Šmulina Theorem, Theorem 15.8, since $L^1(X, \mu)$ is not reflexive then there are bounded sequences in $L^1(X, \mu)$ that do not have weakly convergent subsequences).

Note. The Dunford-Pettis Theorem implies that the condition of uniform integrability for a sequence is fundamental to this problem. So we recall the definition of uniformly integrable from Section 4.6, "Uniform Integrability: The Vitali Convergence Theorem," but state it here for $L^1(X, \mu)$.

Definition. A sequence $\{f_n\}$ in $L^1(X, \mu)$ is *uniformly integrable* over X provided for each $\varepsilon > 0$ there is $\delta > 0$ such that for any measurable set $E \subset X$,

$$\text{if } \mu(E) < \delta \text{ then } \int_E |f_n| \, d\mu < \varepsilon \text{ for all } n \in \mathbb{N}.$$

Note. The next result gives a classification of uniformly integrable in a certain setting.

Proposition 19.10. for a finite measure space $(X, \mathcal{M}, \mu)$ and bounded sequence $\{f_n\}$ in $L^1(X, \mu)$, the following two properties are equivalent:

(i) $\{f_n\}$ is uniformly integrable over X.

(ii) For each $\varepsilon > 0$, there is $M > 0$ such that

$$\int_{\{x \in X \,|\, |f_n(x)| \geq M\}} |f_n| \, d\mu < \varepsilon \text{ for all } n \in \mathbb{N}.$$

Definition. For extended real-valued measurable function f on X and $\alpha > 0$, define the *truncation at level* α of f, denoted $f^{[\alpha]}$, on X by

$$f^{[\alpha]}(x) = \begin{cases} 0 & \text{if } f(x) > \alpha \\ f(x) & \text{if } -\alpha \leq f(x) \leq \alpha \\ 0 & \text{if } f(x) < -\alpha. \end{cases}$$

Note. If $\mu(X) < \infty$ then for $f \in L^1(X, \mu)$ and $\alpha > 0$, we have $f^{[\alpha]} \in L^1(X, \mu)$ and

$$\left| \int_X (f - f^{[\alpha]}) \, d\mu \right| = \left| \int_{\{x \in X \,|\, |f(x)| > \alpha\}} f \, d\mu \right| \leq \int_{\{x \in X \,|\, |f(x)| > \alpha\}} |f| \, d\mu. \qquad (24)$$

Lemma 19.11. For a finite measure space $(X, \mathcal{M}, \mu)$ and bounded uniformly integrable sequence $\{f_n\}$ in $L^1(X, \mu)$, there is a subsequence $\{f_{n_k}\}$ such that for each measurable subset E of X, the sequence of real numbers $\{\int_E f_{n_k} \, d\mu\}$ is Cauchy.

Theorem 19.12. The Dunford-Pettis Theorem.

For a finite measure space $(X,\mathcal{M},\mu)$ and bounded sequence $\{f_n\}$ in $L^1(X,\mu)$, the following two properties are equivalent:

(i) $\{f_n\}$ is uniformly integrable over X.

(ii) Every subsequence of $\{f_n\}$ has a further subsequence that converges weakly in $L^1(X,\mu)$.

Note. We now give a direct condition on a sequence $\{f_n\} \subset L^1(X,\mu)$ which implies the existence of a weakly convergent subsequence.

Corollary 19.13. Let $(X,\mathcal{M},\mu)$ be a finite measure space and $\{f_n\}$ a sequence in $L^1(X,\mu)$ that is dominated by the function $g \in L^1(X,\mu)$ in the sense that

$$|f_n| \le g \text{ a.e. on } E \text{ for all } n \in \mathbb{N}.$$

Then $\{f_n\}$ has a subsequence that converges weakly in $L^1(X,\mu)$.

Note. Royden and Fitzpatrick, 4th edition (2010) includes another corollary to the Dunford-Pettis Theorem. However, it is (for some reason) omitted from the 2018 "Updated Printing" of the book.

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