Chapter 2. Lebesgue Measure

Section 2.1. Introduction

Note. We "weigh" an interval by its length when setting up the Riemann integral. So to generalize the Riemann integral, we desire a way to weigh sets other than intervals. This weight should be a generalization of the length of an interval.

Note. Since we know an open set is a countable union of disjoint open intervals, we would define its "weight" (or "measure") to be the sum of the lengths of the open intervals which compose it.

Note. We want a function m which maps the collection of all subsets of $\mathbb{R}$, that is the power set of the reals $\mathcal{P}(\mathbb{R})$, into $\mathbb{R}^+ \cup \{0, \infty\} = [0, \infty]$. We would like m to satisfy:

1. For any interval I, $m(I) = \ell(I)$ (where $\ell(I)$ is the length of I).
2. For all E on which m is defined and for all $y \in \mathbb{R}$, $m(E + y) = m(E)$. That is, m is *translation invariant*.
3. If $\{E_k\}_{k=1}^{\infty}$ is a sequence of disjoint sets (on each of which, m is defined), then $m(\cupdot E_k) = \sum m(E_k)$. That is, m is *countably additive*.
4. m is defined on $\mathcal{P}(\mathbb{R})$.

Here, and throughout, we use the symbol $\cupdot$ to indicate disjoint union.

Note. We will see in Section 2.6 that there is *not* a function satisfying all four properties. In fact, there is not even a set function satisfying (1), (2), and (4) for which $m\left(\cup_{k=1}^n E_k\right) = \sum_{k=1}^n m(E_k)$ for disjoint E_k (this property is called *finite additivity*). See Theorem 2.18 for details.

Note. It is "unknown" whether m exists satisfying properties (1), (3), and (4) (if we assume the Continuum Hypothesis, then there is *not* such a function).

Note. We will weaken Property (4) and try to find a function defined on as large a set as possible. We will require (by (3)) that our collection of sets, $\mathcal{M}$, on which m is defined, be countably additive and therefore $\mathcal{M}$ will be a σ-algebra.

Problem 2.1. Let m' be a set function defined on a σ-algebra $\mathcal{A}$ with values in $[0, \infty]$. Assume m' is countably additive over countable disjoint collections in $\mathcal{A}$. If A and B are two sets in $\mathcal{A}$ with $A \subset B$, then $m'(A) \leq m'(B)$. This is called *monotonicity*.

Note. Another property of measure is the following.

Problem 2.3. Let $\{E_k\}_{k=1}^\infty$ be a countable collection of sets in a σ-algebra $\mathcal{A}$ on which a countably additive measure m' is defined. Then $m'\left(\cup_{k=1}^\infty E_k\right) \leq \sum_{k=1}^\infty m'(E_k)$. This is called *countable subadditivity*.