Section 2.2. Lebesgue Outer Measure

Note. We now introduce a type of measure with an eye towards the requirement that $m(I) = \ell(I)$ for intervals (at least, for open intervals).

Definition. Let $A \subset \mathbb{R}$ and let $\{I_n\}$ represent a countable collection of bounded open intervals such that $A \subset \cup I_n$. The *outer measure* of A is

$$m^*(A) = \inf_{A \subset \cup I_n} \left\{ \sum_{n=1}^{\infty} \ell(I_n) \right\}$$

where the infimum is taken over all such open interval coverings of A.

Note. Since $m^*(A)$ is defined as an infimum, then $m^*(A)$ is defined for every $A \in \mathcal{P}(\mathbb{R})$. $m^*(\varnothing) = 0$ and if A is finite in cardinality then $m^*(A) = 0$. Also, if $A \subset B$ then $m^*(A) \leq m^*(B)$ (i.e., m^* satisfies monotonicity). Royden and Fitzpatrick justify this with a brief comment. Here's a detailed proof.

Lemma 2.2.A. Outer measure is monotone. That is, if $A \subset B$ then $m^*(A) \leq m^*(B)$.

Note. The following result shows that we are on the right track for the condition $m^*(I) = \ell(I)$. Here and throughout we have that the length of an unbounded interval is ∞.

Proposition 2.1. The outer measure of an interval is its length.

Note. Now for desired property (2) (translation invariance).

Proposition 2.2. Outer measure is translation invariant; that is, for any set A and number y, $m^*(A+y) = m^*(A)$.

Note. Now for a result *related to* desired property (3), but not property (3) itself.

Proposition 2.3. Outer measure is countably subadditive. That is, if $\{E_k\}_{k=1}^{\infty}$ is any countable collection of sets then

$$m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} m^*(E_k).$$

Note. In conclusion, outer measure m^* satisfies:

1. For any interval I, $m^*(I) = \ell(I)$ (Proposition 2.1).
2. m^* is translation invariant (Proposition 2.2).
3. m^* is countably *sub*additive (Proposition 2.3).
4. m^* is defined on $\mathcal{P}(\mathbb{R})$.

So m^* is close to satisfying the desired four properties. However, we must have countable additivity so that we can use the measure for the basis of integration.

In the next section we put a condition on a set (called the "Carathéodory splitting condition") and consider the collection of all such sets satisfying this condition. We will see that m^* is countably additive on this collection of sets.

Exercise 2.5. $[0, 1]$ is not countable.

Note. The following result is further motivation for studying G_δ sets.

Exercise 2.7. For any bounded set E, there is a G_δ set G for which $E \subset G$ and $m^*(G) = m^*(E)$. Set G is called the *measurable cover* of E (see Theorem 3.1 of the supplemental notes to Section 2.3).

Revised: 8/29/2020