Section 2.3. The σ-Algebra of Lebesgue Measurable Sets

Note. In Theorem 2.18 we will see that there are disjoint sets A and B such that

$$m^*(A \cup B) < m^*(A) + m^*(B)$$

and so m^* is not countably additive (it is not even finitely additive). We must have countable additivity in our measure in order to use it for integration. To accomplish this, we restrict m^* to a smaller class than $\mathcal{P}(\mathbb{R})$ on which m^* *is* countably additive. As we will see, the following condition will yield a collection of sets on which m^* is countably additive.

Definition. Any set E is (*Lebesgue*) *measurable* if for all $A \subset \mathbb{R}$,

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c).$$

This is called the *Carathéodory splitting condition.*

Note. It certainly isn't clear that the Carathéodory splitting condition leads to countable additivity, but notice that it does involve the sum of measures of two disjoint sets ($A \cap E$ and $A \cap E^c$).

Note. The Carathéodory splitting condition, in a sense, requires us to "check" all $\aleph_2$ subsets A of $\mathbb{R}$ in order to establish the measurability of a set E. It is shown in Problem 2.20 that if $m^*(E) < \infty$, then E is measurable if and only if $m^*((a,b)) = m^*((a,b) \cap E) + m^*((a,b) \cap E^c)$. This is a significant result, since there are "only" $\aleph_1$ intervals of the form (a,b).

Note. Recall that we define a function to be Riemann integrable if the upper Riemann integral equals the lower Riemann integral. We might expect to define "measurable" in a similar way. The following discussion of inner measure is based on *Real Analysis* by Bruckner, Bruckner, and Thomson, Prentice Hall, (1997).

Definition. Let $\lambda((a,b)) = \ell((a,b)) = b-a$, with $\lambda((a,b)) = \infty$ if (a,b) is unbounded. For open set $G = \cup_{k=1}^{\infty} I_k$ (where the I_k are open intervals), define $\lambda(G) = \displaystyle\sum_{k=1}^{\infty} \lambda(I_k)$, and if $G = \varnothing$ define $\lambda(G) = 0$.

Definition. Let E be a closed and bounded set with $a = \text{glb}(E)$ and $b = \text{lub}(E)$. Define $\lambda(E) = b - a - \lambda((a,b) \setminus E)$.

Note. For closed E, $(a,b) \setminus E = (a,b) \cap E^c$ is open. Also, $\lambda(E) + \lambda((a,b) \setminus E) = b - a$.

Definition. Let $E \subset \mathbb{R}$. Then

$$\lambda^*(E) = \inf\{\lambda(G) \mid E \subset G, G \text{ is open}\}$$

is the (Lebesgue) *outer measure* of E.

Note. This definition is equivalent to Royden's definition of outer measure.

Definition. Let $E \subset \mathbb{R}$. Then $\lambda_*(E) = \sup\{\lambda(F) \mid F \subset E, F \text{ is compact}\}$ is the *inner measure* of E. If $E = \varnothing$, define $\lambda_*(E) = 0$.

Note. We have seen that outer measure λ^* is: translation invariant, monotone, $\lambda^*(I) = \ell(I)$ for all intervals, and is countably subadditive. We can similarly show that inner measure λ_* is: translation invariant, monotone, $\lambda_*(I) = \ell(I)$, and is countably superadditive:

$$\lambda_* \left(\dot\cup_{k=1}^{\infty} E_k\right) \geq \sum_{k=1}^{\infty} \lambda_*(E_k).$$

Note. In Royden and Fitzpatrick's Problem 2.7 and Theorem 2.11 (with a few additional details) we have that for all $E \subset \mathbb{R}$: (1) There is a G_δ set G such that $E \subset G$ and $\lambda^*(E) = \lambda^*(G)$, and (2) there is an F_σ set F such that $F \subset E$ and $\lambda_*(E) = \lambda_*(F)$. Set G is called the *outer approximation* (*outer content* or *measurable cover*) of E and set F is called the *inner approximation* (*inner content* or *measurable kernel*) of set E.

Note. The following is the definition of "measurable" which parallels the definition of "Riemann integrable."

Definition. Let $E \subset \mathbb{R}$ be bounded. If $\lambda_*(E) = \lambda^*(E)$ then E is said to be (*Lebesgue*) *measurable* with (*Lebesgue*) *measure* $m(E) = \lambda^*(E)$. If E is unbounded, then E is *measurable* if $E \cap I$ is measurable for every finite interval I and $m(E) = \lambda^*(E)$.

Note. Since this definition of measurable is based only on bounded sets and we can establish the relationship for all bounded $E \subset \mathbb{R}$ (Bruckner, Bruckner, Thomson):

$$\lambda_*(E) = b - a + \lambda^*([a, b] \setminus E)$$

(where $a = \mathrm{glb}(E)$ and $b = \mathrm{lub}(E)$), then we see that inner measure is ultimately dependent only on outer measure. This is how Royden and Fitzpatrick are able to develop measure theory without reference to inner measure.

Note. One can show that the Carathéodory splitting condition on set E implies that $\lambda^*(E) = \lambda_*(E)$ AND that $\lambda^*(E) = \lambda_*(E)$ implies the Carathéodory splitting condition on set E. Therefore the splitting condition approach is equivalent to the inner/outer measure approach.

Note. Henri Lebesgue (1875–1941) was the first to crystallize the ideas of measure and the integral studied in Part 1 of our class. In his doctoral dissertation, *Intégrale, Longueur, Aire* ("Integral, Length, Area") of 1902, he presented the definitions of inner and outer measure equivalent to the approach above of Bruckner and Bruckner. His definition of "measurable" is the same as that given above (notice the similarity to the definition of Riemann integral which is based on upper and lower Riemann integrals). Lebesgue published his results in 1902, with the same title as his dissertation, in *Annali di Matematica Pura ed Applicata*, Series 3, VII(4), 231–359, available online (in French, or course). Carathéodory introduced his splitting condition in 1914. His approach to outer measure and measurability in a more abstract setting is explored in Part 3 of the text (in particular, Section

20.4). His results appeared in Über das lineare Mass von Punktmengen—eine Verallgemeinerung des Längenbegriffs ["About the linear measure of sets of points—a generalization of the concept of length"] *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse* ["News of the Society of Sciences in Göttingen, Mathematics and Physical Class"] (1914), 404–426. These websites were last accessed 3/11/2016. We now return to Royden and Fitzpatrick's approach...

Note 2.3.A. Since $A \cap E$ and $A \cap E^c$ are disjoint, then by the countable subadditivity of outer measure m^*,

$$m^*(A) \leq m^*(A \cap E) + m^*(A \cap E^c).$$

Therefore to show that E is measurable, we need only show that for all $A \subset \mathbb{R}$,

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c).$$

(This appears on page 35 of Royden and Fitzpatrick.) This holds trivially for all A where $m^*(A) = \infty$, so we see that to show E is measurable we need only to consider sets A of finite outer measure. Also, if E satisfies this inequality, then so does E^c. In addition $E = \varnothing$ and $E = \mathbb{R}$ satisfy this inequality and so are measurable. Our goal in this section is to show that all sets satisfying the splitting condition form a σ-algebra ("the σ-algebra of Lebesgue measurable sets").

Proposition 2.4. If $m^*(E) = 0$, then E is measurable.

Note. We use the notation E^c to indicate $\mathbb{R} \setminus E$. The notation $\tilde{E}$ was used in the more general setting where the universal set (usually denoted X) was not necessarily $\mathbb{R}$.

Note. We denote the set of all Lebesgue measurable sets as $\mathcal{M}$. We have seen that $\mathcal{M}$ includes $\varnothing$, $\mathbb{R}$, and all sets of outer measure 0. Also, $\mathcal{M}$ is closed under complements by Note 2.3.A.

Proposition 2.5. The union of a finite collection of measurable sets is measurable.

Note. Proposition 2.5, along with the previous observation, implies that $\mathcal{M}$ is an algebra of sets. The following three results establish that $\mathcal{M}$ is in fact a σ-algebra and that m^* is countably additive on $\mathcal{M}$.

Proposition 2.6. Let $A \subset \mathbb{R}$ and let $\{E_k\}_{k=1}^n$ be a finite disjoint collection of measurable sets. Then

$$m^* \left(A \cap [\cup_{k=1}^n E_k]\right) = \sum_{k=1}^n m^*(A \cap E_k),$$

In particular, when $A = \mathbb{R}$ we see that m^* is finite additive on $\mathcal{M}$.

Proposition 2.7. The union of a countable collection of measurable sets is measurable.

Note 2.3.B. Since $\mathcal{M}$ is closed under compliments and closed under countable unions, then $\mathcal{M}$ is a σ-algebra.

Note. We jump ahead a bit to prove countable additivity of m^* on $\mathcal{M}$.

Proposition 2.13. (From Section 2.5.) If $\{E_k\}_{k=1}^{\infty} \subset \mathcal{M}$ and the E_k are pairwise disjoint, then

$$m^*\left(\cup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} m^*(E_k).$$

Note. We know that $m^*(I) = \ell(I)$ for each interval, but we have not yet shown that intervals are measurable.

Proposition 2.8. Every interval is measurable.

Note. We now see that $\mathcal{M}$ is a σ-algebra containing all open intervals. Therefore $\mathcal{M}$ contains all Borel sets.

Proposition 2.10. The translate of a measurable set is measurable.

Note. In conclusion, outer measure m^* on σ-algebra $\mathcal{M}$ satisfies:

1. For any interval I, $m^*(I) = \ell(I)$ (Proposition 2.1).
2. m^* is translation invariant (Proposition 2.2).
3. m^* is countably additive on $\mathcal{M}$ (Proposition 2.13).
4. $\mathcal{M} \neq \mathcal{P}(\mathbb{R})$, as we will see in Section 2.6, "Nonmeasurable Sets."

In Section 2.5, "Countable Additivity, Continuity, and the Borel-Cantelli Lemma," we define *Lebesgue measure* as m^* restricted to $\mathcal{M}$.