Section 2.4. Outer and Inner Approximation of Lebesgue Measurable Sets

Note. In this section we give several conditions on $E \subset \mathbb{R}$ which are equivalent to the measurability of E. In the process, we "approximate" measurable sets with more familiar sets.

Lemma 2.4.A. The Excision Property.
If A is measurable and $m^*(A) < \infty$ and $A \subset B$ then $m^*(B \setminus A) = m^*(B) - m^*(A)$.

Note. You showed in Problem 2.7 that for any *bounded* set E, there is a G_δ set G such that $E \subset G$ and $m^*(G) = m^*(E)$. We see in the following theorem that a similar result holds for measurable sets, and also that there is an analogous result for an F_σ subset of E.

Theorem 2.11. Let $E \subset \mathbb{R}$. Then each of the following are equivalent to the measurability of E:

1. For each $\varepsilon > 0$, there is an open set $\mathcal{O}$ containing E for which $m^*(\mathcal{O} \setminus E) < \varepsilon$.
2. There is a G_δ set G containing E for which $m^*(G \setminus E) = 0$.
3. For each $\varepsilon > 0$, there is a closed set F contained in E for which $m^*(E \setminus F) < \varepsilon$.
4. There is an F_σ set F contained in E for which $m^*(E \setminus F) = 0$.

Note. The G_δ set G of Theorem 2.11 is the *outer approximation* of measurable E and the F_σ set F is called the *inner approximation.* Notice that Theorem 2.11 tells us that we can "approximate" a measurable set E with both a G_δ set G and an F_σ set F. The approximation is done in the sense of measure as spelled out in Theorem 2.11. Notice that F and G are Borel and so measurable (see the Note after Proposition 2.8 in the class notes). Therefore by countable additivity (Proposition 2.13), $m^*(E \cup (G \setminus E)) = m^*(E) + m^*(G \setminus E) = m^*(G)$ and so $m^*(E) = m^*(G)$. Similarly, $m^*(E) = m^*(F)$. We can now conclude that: Every measurable set is "almost" an F_σ set and "almost" a G_δ set. More precisely, every measurable set (a) differs from an F_σ subset by a set of measure zero, and (b) differs from a G_δ superset by a set of measure zero.

Note. In the study of inner and outer measure (see the supplement to the notes for Section 2.3) we introduce the *measurable cover* G and *measurable kernal* F of (not necessarily measurable) set E as G_δ set G and F_σ set F such that $F \subset E \subset G$ where the inner measure of F equals the inner measure of E and the outer measure of G equals the outer measure of E. Notice that these ideas do not involve measurable sets! In this general setting, we do not necessarily have an equality of the inner measure of F and the outer measure of G. In conclusion, "inner approximation" and "outer approximation" are associated with measurable sets, and "measurable kernal" and "measurable cover" are associated with any set of real numbers.

Definition. The *symmetric difference* of sets A and B, denoted $A\Delta B$, is $A\Delta B = (A \setminus B) \cup (B \setminus A)$.

Note. If A and B are measurable sets of real numbers then, since $A \setminus B$ and $B \setminus A$ are disjoint, by countable additivity (Proposition 2.13),

$$m^*(A\Delta B) = m^*(A \setminus B) + m^*(B \setminus A).$$

Theorem 2.12. Let $E \in \mathcal{M}$, $m^*(E) < \infty$. Then for each $\varepsilon > 0$, there is a finite disjoint collection of open intervals $\{I_k\}_{k=1}^n$ for which, if $\mathcal{O} = \cup_{k=1}^n I_k$, then

$$m^*(E\Delta\mathcal{O}) = m^*(E \setminus \mathcal{O}) + m^*(\mathcal{O} \setminus E) < \varepsilon.$$

Note. The text says that this result shows that finite measurable sets are "nearly" a finite union of disjoint open intervals. In general, when we can approximate a set (or, later, a function) to within an arbitrary given ε, we will use the word "nearly."

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