Section 2.5. Countable Additivity, Continuity, and the Borel-Cantelli Lemma

Note. We finally define Lebesgue measure. We have already seen countable additivity in Section 2.3.

Definition. Outer measure $m^*$ on $\mathcal{M}$ is called Lebesgue measure and is denoted $m$.

Note. Based on what has been established, we have:

Theorem 2.14. The set function Lebesgue measure, $m$, defined on the $\sigma$-algebra $\mathcal{M}$ of Lebesgue measurable sets, assigns length to any interval, is translation invariant, and is countably additive.

Theorem 2.15. Measure is Continuous.

Lebesgue measure satisfies:

(i) If $\{A_k\}_{k=1}^{\infty}$ is an ascending collection of measurable sets (i.e., $A_k \subset A_{k+1}$), then

$$m(\bigcup_{k=1}^{\infty} A_k) = m(\lim_{k \to \infty} A_k) = \lim_{k \to \infty} m(A_k).$$

(ii) If $\{B_k\}_{k=1}^{\infty}$ is a descending collection of measurable sets (i.e., $B_k \supset B_{k+1}$) and $m(B_1) < \infty$, then

$$m(\bigcap_{k=1}^{\infty} B_k) = m(\lim_{k \to \infty} B_k) = \lim_{k \to \infty} m(B_k).$$
Problem 2.25. Show that the assumption that \( m(B_1) < \infty \) is necessary in part (ii) of Theorem 2.15.

The Idea. Consider \( B_k = [k, \infty) \).

Definition. Let \( E \in \mathcal{M} \). A property holds almost everywhere on \( E \) if there is \( E_0 \subset E \) where \( m(E_0) = 0 \) and the property holds on \( E \setminus E_0 \). We also say the property holds for almost all \( x \in E \).

Note. We can restate the Riemann-Lebesgue Theorem using the new verbiage: A bounded function \( f \) defined on \([a, b]\) is Riemann integrable on \([a, b]\) if and only if \( f \) is continuous almost everywhere on \([a, b]\).

The Borel-Cantelli Lemma.

Let \( \{E_k\}_{k=1}^{\infty} \) be a countable collection of measurable sets for which \( \sum m(E_k) < \infty \). Then almost all \( x \in \mathbb{R} \) belong to at most finitely many of the \( E_k \)'s.

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