Section 2.5. Countable Additivity, Continuity, and the Borel-Cantelli Lemma

Note. In this section, we finally define Lebesgue measure. We have already seen that outer measure is countable additive on the σ-algebra of Lebesgue measurable sets in Section 2.3 (see Proposition 2.13).

Definition. Outer measure m^* on $\mathcal{M}$ (the σ-algebra of Lebesgue measurable sets) is called *Lebesgue measure* and is denoted m.

Note. Based on what has been established, we have:

Theorem 2.14. The set function Lebesgue measure, m, defined on the σ-algebra $\mathcal{M}$ of Lebesgue measurable sets, assigns length to any interval, is translation invariant, and is countably additive.

Note. Recall that a function f is continuous at a point x_0 in its domain, if the limit as x approaches x_0 of f is $f(x_0)$. That is, $\lim_{x \to x_0} f(x) = f\left(\lim_{x \to x_0} x\right) = f(x_0)$. So we can pass limits in and out of continuous functions. This is the motivation for the name of the following theorem.

Theorem 2.15. Measure is Continuous.

Lebesgue measure satisfies:

(i) If $\{A_k\}_{k=1}^\infty$ is an ascending collection of measurable sets (i.e., $A_k \subset A_{k+1}$), then $m(\cup_{k=1}^\infty A_k) = m(\lim_{k\to\infty} A_k) = \lim_{k\to\infty} m(A_k)$.

(ii) If $\{B_k\}_{k=1}^\infty$ is a descending collection of measurable sets (i.e., $B_k \supset B_{k+1}$) and $m(B_1) < \infty$, then $m(\cap_{k=1}^\infty B_k) = m(\lim_{k\to\infty} B_k) = \lim_{k\to\infty} m(B_k)$.

Note. The assumption of finite measure of B_1 in part (ii) of Theorem 2.15 is necessary, as is to be shown in Problem 2.25.

Definition. Let $E \in \mathcal{M}$. A property holds *almost everywhere on* E if there is $E_0 \subset E$ where $m(E_0) = 0$ and the property holds on $E \setminus E_0$. We also say the property holds for *almost all* $x \in E$.

Note. We can restate the Riemann-Lebesgue Theorem using the new verbiage: A bounded function f defined on $[a, b]$ is Riemann integrable on $[a, b]$ if and only if f is continuous almost everywhere on $[a, b]$.

The Borel-Cantelli Lemma.

Let $\{E_k\}_{k=1}^\infty$ be a countable collection of measurable sets for which $\sum_{k=1}^\infty m(E_k) < \infty$. Then almost all $x \in \mathbb{R}$ belong to at most finitely many of the E_k's.

Note. Royden and Fitzpatrick play up the importance of countable additivity in the following quote from page 46:

> "In our forthcoming study of Lebesgue integration it will be apparent that it is the countable additivity of Lebesgue measure that provides the Lebesgue integral with its decisive advantage over the Riemann integral."

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