# **Section 2.6.** Nonmeasurable Sets (Royden and Fitzpatrick, 4th Edition)

**Note.** Your humble instructor prefers the presentation on nonmeasurable sets given in Royden's 3rd edition of *Real Analysis* (given in the notes as a supplement to this section). The "construction" presented here is just an alternative approach to the standard Vitali construction given in the supplement.

**Lemma 2.16.** Let $E$ be a bounded measurable set of real numbers. Suppose there is a bounded countably infinite set of real numbers $\Lambda$ for which the collection of translates of $E$, $\{E + \lambda\}_{\lambda \in \Lambda}$ is disjoint. Then $m(E) = 0$.

**Definition.** For any nonempty set $E$ of real numbers, two points in $E$ are *rationally equivalent* if their difference is rational.

**Note.** Rational equivalence is an equivalence relation (reflexive, symmetric, and transitive) or any nonempty set of real numbers. Recall that an equivalence relation on a set partitions the set into equivalence classes.

**Definition.** Let $E$ be a nonempty set of real numbers with rational equivalence as an equivalence relation on $E$. A *choice set* is a set $\mathcal{C}_E \subset E$ such that $\mathcal{C}_E$ contains exactly one element of each equivalence class under rational equivalence.

**Note.** A choice set exists by the Axiom of Choice (see page 5). Notice that the choice set $\mathcal{C}_E$ satisfies the following two properties:

**(i)** the difference of two points in $\mathcal{C}_E$ is irrational,

**(ii)** for each point $x \in E$, there is a point $c \in \mathcal{C}_E$ for which $x = c + q$ for some $q \in \mathbb{Q}$.

Notice that the first property implies that:

$$\text{For any set } \Lambda \subset \mathbb{Q}, \text{ the collection } \{\mathcal{C}_E + \lambda\}_{\lambda \in \Lambda} \text{ is disjoint.}$$

**Theorem 2.17. The Vitali Construction of a Nonmeasurable Set.** Any set $E$ of real numbers with positive outer measure contains a subset that fails to be measurable.

**Theorem 2.18.** There are disjoint sets of real numbers $A$ and $B$ for which

$$m^*(A \cup B) < m^*(A) + m^*(B).$$

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