Section 2.6. Nonmeasurable Sets (Royden's 3rd Edition)

Note. In this section, we follow the technique of Royden's 3rd edition to construct a nonmeasurable set. This is equivalent to the "construction" given in the 4th edition and is based on Vitali's original proof... however, I find the proof from the 3rd edition more tangible.

Definition. Let $x, y \in [0, 1)$. Define

$$x \mathring{+} y = \begin{cases} x + y & \text{if } x + y < 1 \\ x + y - 1 & \text{if } x + y \geq 1. \end{cases}$$

Lemma 2.6.A. Let $E \subset [0, 1)$ and $E \in \mathcal{M}$. Then for all $y \in [0, 1)$, $E \mathring{+} y$ is measurable and $m(E \mathring{+} y) = m(E)$.

Definition. Let $x, y \in [0, 1)$. Then x and y are said to be *rationally equivalent*, denoted $x \sim y$, if $x - y \in \mathbb{Q}$. Notice that $\sim$ is an equivalence relation on $[0, 1)$ and so $\sim$ partitions $[0, 1)$ into equivalence classes.

Definition. Let $\mathcal{F}$ be a nonempty family of nonempty sets. A *choice function* f on $\mathcal{F}$ is a function f from $\mathcal{F}$ to $\cup_{F \in \mathcal{F}} F$ with the property that for each set $F \in \mathcal{F}$, $f(F)$ is a member of F.

Example. Let $\mathcal{F} = \{A, B, C\}$ where $A = \{0, 1, 2\}$, $B = \{\text{red}, \text{blue}, \text{green}\}$, and $C = \{\text{truth}, \text{beauty}, \text{love}\}$. Then an example of a choice function on $\mathcal{F}$ is $f(A) = 0$, $f(B) = \text{blue}$, and $f(C) = \text{love}$.

The Axiom of Choice. (Ernst Zermelo, 1908.)
Let $\mathcal{F}$ be a nonempty collection of nonempty sets. Then there is a choice function on $\mathcal{F}$.

Note. The Axiom of Choice guarantees the existence of a choice function, but it does not say *how* the choice function is defined. For this reason (among others, including some weird implications of the Axiom of Choice) some mathematicians reject the Axiom of Choice. All mathematicians are conscious of when they use it and clearly make note of this. We note that it is used in the following "construction."

Note. By the Axiom of Choice, there exists a representative of each equivalence class of $\sim$ from the partitioning of $[0, 1)$ into equivalence classes by rational equivalence. Let P be a set consisting of exactly one representative from each equivalence class.

Note. We will show that P is not measurable (i.e., $P \notin \mathcal{M}$) by contradiction.

Theorem 2.6.B. Set P is not measurable.

Note. The idea here is that the $\{P_i\}_{i=0}^{\infty}$ (as defined in the proof of Theorem B) is a disjoint collection of sets, all of the same "size" (i.e., measure... assuming their measurability; each is a $\overset{\circ}{+}$ translate of P) which partition $[0, 1)$. We are lead to a contradiction about the Lebesgue measure of $[0, 1)$ under the only two possible options: $m(P) = 0$ or $m(P) > 0$. This basic idea is behind the only paper your humble instructor has which remotely involves measure theory. The paper is "Translation Invariance and Finite Additivity in a Probability Measure on the Natural Numbers" by R. Gardner and R. Price, *International Journal of Mathematics and Mathematical Sciences*, **29**(10) (2002), 585–589. The paper is available online.

Philosophical Note. We have used the Axiom of Choice to create P. This means that we have *no idea* what P "looks like" (i.e., what is in P and what is not— is $1/2 \in P$?)! A reasonable question is "What is the outer measure of P: $m^*(P) =$?" This is defined since m^* is defined on $\mathcal{P}(\mathbb{R})$. We know $m^*(P) \neq 0$, or else we would have $P \in \mathcal{M}$ (which is not the case). However, we cannot answer this question since we have no idea what is in P! Again, this is an objection some mathematicians have to the Axiom of Choice.

Theorem 2.18. (From the 4th Edition.)

There are disjoint sets of real numbers A and B for which $m^*(A \cup B) < m^*(A) + m^*(B)$.

Note. We have used the Axiom of Choice to show the existence of a nonmeasurable set. Is there another way to show the existence of a nonmeasurable set? NO! In 1964, Robert Solovay showed that we cannot prove $\mathcal{M} \neq \mathcal{P}(\mathbb{R})$ without the Axiom of Choice [A Model of Set Theory in which Every Set of Reals is Lebesgue Measurable, *Annals of Mathematics* **92**(1970), 1–56]. (A copy of this work can currently be viewed at Researchgate website; accessed 9/26/2020.) This work actually lives in the realm of set theory and involves manipulations of the standard Zermelo-Fraenkel axioms of set theory with the Axiom of Choice (ZFC). For more details, see "How Good is Lebesgue Measure?" by Krzysztof Ciesielski, *The Mathematical Intelligencer* **11**(2), 1989, 54–58.

Problem A. Show that if $E \in \mathcal{M}$ and $E \subset P$, then $m(E) = 0$. HINT: Let $E_i = E \stackrel{\circ}{+} r_i$, where $\{r_i\}_{i=1}^{\infty} = \mathbb{Q} \cap [0,1)$. Then $\{E_i\}_{i=1}^{\infty}$ is a disjoint sequence of measurable sets and $m(E_i) = m(E)$. Therefore $\sum m(E_i) = m(\cup E_i) \leq m([0,1))$.

Problem B. Show that if A is any set with $m^*(A) > 0$, then there is a nonmeasurable set $E \subset A$. HINT: If $A \subset (0,1)$, let $E_i = A \cap P_i$. The measurability of E_i implies $m(E_i) = 0$, while $\sum m^*(E_i) \geq m^*(A) > 0$.

Problem C. Give an example $\{E_i\}_{i=1}^{\infty}$ of a disjoint sequence of sets and $m^*(\cup E_i) < \sum m^*(E_i)$. Use set P and explain.

Note. Giuseppe Vitali was born in Italy in 1875. He attended the University of Bologna for two years and graduated from Scuola Normale Superiore in Pisa in 1899. He received a teaching diploma and from 1904 to 1923 he taught secondary school. In 1923 he started at the University of Modena, moved to University of Padua in 1925, and became chair of mathematics at the University of Bologna in 1930. Vitali was an analyst and did his most significant work between 1900 and 1908. In particular, he gave the first example of a nonmeasurable set in 1905 (in *Sul problema della misura dei gruppi di punti di una retta*, Bologna, Tip. Gamberini e Parmeggiani [1905]). He also characterized absolutely continuous functions as antiderivatives of Lebesgue integrable functions in 1905 (this is our Theorem 6.11), and introduced "Vitali coverings" in 1908 (we'll see this in Section 6.2). This information and the following photo is from the MacTutor History of Mathematics archive (accessed 10/9/2016).

Giuseppe Vitali (August 26, 1875 – February 29, 1932)

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