Section 2.7. The Cantor Set and the Cantor-Lebesgue Function

Note. In this section, we define the Cantor set which gives us an example of an uncountable set of measure zero. We use the Cantor-Lebesgue Function to show there are measurable sets which are not Borel; so $\mathcal{B} \subsetneq \mathcal{M}$. The supplement to this section gives these results based on cardinality arguments (but the supplement does not address the Cantor-Lebesgue Function).

Definition. Let $I = [0,1]$. We iteratively remove the "open middle one-third" of closed subintervals of I as follows. We remove:

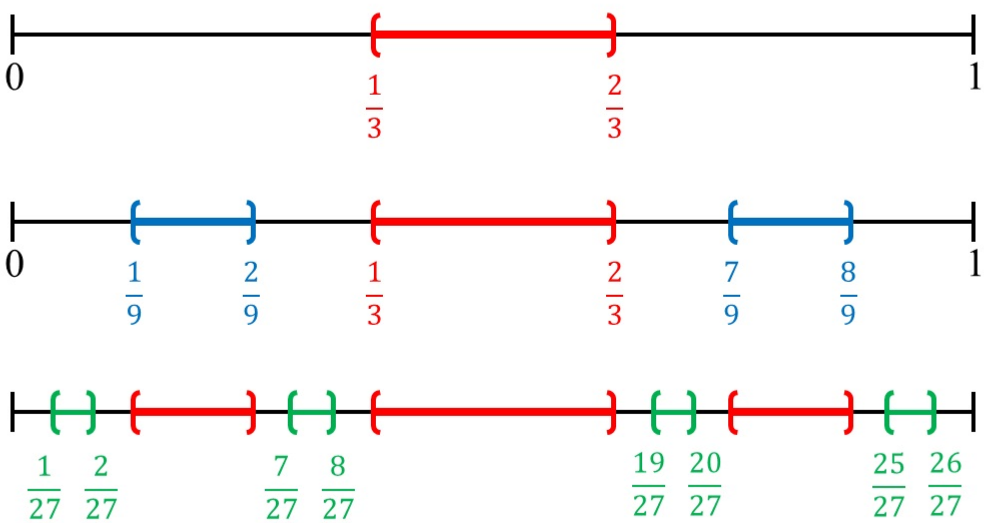
$$\begin{aligned}
\mathcal{O}_1 &= \left(\tfrac{1}{3}, \tfrac{2}{3}\right) \\
\mathcal{O}_2 &= \left(\tfrac{1}{9}, \tfrac{2}{9}\right) \cup \left(\tfrac{7}{9}, \tfrac{8}{9}\right) \\
\mathcal{O}_3 &= \left(\tfrac{1}{27}, \tfrac{2}{27}\right) \cup \left(\tfrac{7}{27}, \tfrac{8}{27}\right) \cup \left(\tfrac{19}{27}, \tfrac{20}{27}\right) \cup \left(\tfrac{25}{27}, \tfrac{26}{27}\right) \\
\mathcal{O}_4 &= \left(\tfrac{1}{81}, \tfrac{2}{81}\right) \cup \left(\tfrac{7}{81}, \tfrac{8}{81}\right) \cup \left(\tfrac{19}{81}, \tfrac{20}{81}\right) \cup \left(\tfrac{25}{81}, \tfrac{26}{81}\right) \cup \left(\tfrac{55}{81}, \tfrac{56}{81}\right) \cup \left(\tfrac{61}{81}, \tfrac{62}{81}\right) \cup \left(\tfrac{73}{81}, \tfrac{74}{81}\right) \cup \left(\tfrac{79}{81}, \tfrac{80}{81}\right) \\
&\vdots \\
\mathcal{O}_k &= 2^{k-1} \text{ open intervals of total length } \tfrac{2^{k-1}}{3^k} \\
&\vdots
\end{aligned}$$

then we get:

$$
\begin{aligned}
C_1 &= \left[0, \tfrac{1}{3}\right] \cup \left[\tfrac{2}{3}, 1\right] \\
C_2 &= \left[0, \tfrac{1}{9}\right] \cup \left[\tfrac{2}{9}, \tfrac{1}{3}\right] \cup \left[\tfrac{2}{3}, \tfrac{7}{9}\right] \cup \left[\tfrac{8}{9}, 1\right] \\
C_3 &= \left[0, \tfrac{1}{27}\right] \cup \left[\tfrac{2}{27}, \tfrac{1}{9}\right] \cup \left[\tfrac{2}{9}, \tfrac{7}{27}\right] \cup \left[\tfrac{8}{27}, \tfrac{1}{3}\right] \cup \left[\tfrac{2}{3}, \tfrac{19}{27}\right] \cup \left[\tfrac{20}{27}, \tfrac{7}{9}\right] \cup \left[\tfrac{8}{9}, \tfrac{25}{27}\right] \cup \left[\tfrac{26}{27}, 1\right] \\
&\vdots \\
C_k &= 2^k \text{ closed intervals of total length } \tfrac{2^k}{3^k} \\
&\vdots
\end{aligned}
$$

The *Cantor set* is $\mathbf{C} = \cap_{k=1}^{\infty} C_k$.

Note. The first three steps in the creation of the Cantor set look like:



Note. We also have that $\mathbf{C} = I \setminus \cup_{k=1}^{\infty} \mathcal{O}_k$ and so $I = \mathbf{C} \,\dot\cup\, (\dot\cup_{k=1}^{\infty} \mathcal{O}_k)$.

Proposition 2.19. The Cantor set $\mathbf{C}$ is a closed, uncountable set of measure zero.

Note. In the supplement to this section, it is argued that $|\mathcal{M}| = \aleph_2$. This is based on the fact that $|\mathbf{C}| = \aleph_1$ and so $|\mathcal{P}(\mathbf{C})| = \aleph_2$. Each subset of $\mathbf{C}$ is of outer measure zero by monotonicity (Lemma 2.2.A) and so is measurable by Proposition 2.4.

Note. Exercise 2.39 describes the construction of a "fat" Cantor set of measure $1 - \alpha$ where $\alpha \in (0, 1)$. In Exercise 2.40, it is stated that an open set can have a boundary of positive measure (the open set is the complement of any fat Cantor set and the boundary is then the fat Cantor set itself). Notice how unintuitive this is since we know that an open set consists of a countable number of disjoint open intervals (so the boundary would seem to be simply the endpoints of these intervals, but it must be more complicated).

Definition. A real-valued function f that is defined on a set of real numbers is *increasing* if $f(u) \leq f(v)$ whenever $u \leq v$. It is *strictly increasing* if $f(u) < f(v)$ whenever $u < v$.

Definition. Let $k \in \mathbb{N}$. Define increasing function φ on $[0, 1] \setminus C_k$, which consists of $2^k - 1$ disjoint open intervals, to take on the values $\{1/2^k, 2/2^k, 3/2^k, \ldots, (2^k - 1)/2^k\}$. For example:

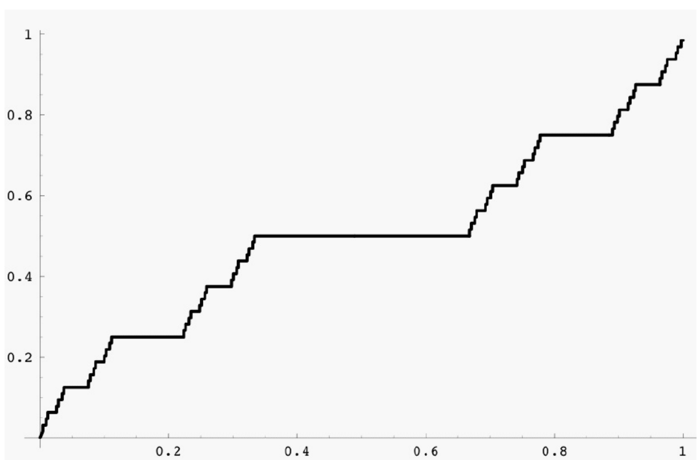
$$k = 1 \text{ implies } \varphi(x) = 1/2 \text{ if } x \in (1/3, 2/3),$$

$$k = 2 \text{ implies } \varphi(x) = \begin{cases} 1/4 \text{ if } x \in (1/9, 2/9) \\ 2/4 \text{ if } x \in (3/9, 6/9) = (1/3, 2/3) \\ 3/4 \text{ if } x \in (7/9, 8/9) \end{cases}$$

$$k = 3 \text{ implies } \varphi(x) = \begin{cases} 1/8 \text{ if } x \in (1/27, 2/27) \\ 2/8 \text{ if } x \in (3/27, 6/27) = (1/9, 2/9) \\ 3/8 \text{ if } x \in (7/27, 8/27) \\ 4/8 \text{ if } x \in (9/27, 18/27) = (1/3, 2/3) \\ 5/8 \text{ if } x \in (19/27, 20/27) \\ 6/8 \text{ if } x \in (21/27, 24/27) = (7/9, 8/9) \\ 7/8 \text{ if } x \in (25/27, 26/27). \end{cases}$$

In this way, φ is defined on $\mathcal{O} = \cup_{k=1}^{\infty} \mathcal{O}_k = [0, 1] \setminus \mathbf{C}$. For $x \in \mathbf{C}$, define $\varphi(x) = 0$ if $x = 0$ and $\varphi(x) = \sup\{\varphi(t) \mid t \in \mathcal{O} \cap [0, x)\}$ if $x \in \mathbf{C} - \setminus\{0\}$. φ is the *Cantor-Lebesgue function* (or simply the *Cantor function*).

Note. A nice numerical graph of the Cantor-Lebesgue function is:



This image is from mathproblems123.wordpress.com.

Proposition 2.20. The Cantor-Lebesgue function φ is an increasing continuous function that maps $[0, 1]$ onto $[0, 1]$. Its derivative exists on the open set $\mathcal{O} = [0, 1] \setminus \mathbf{C}$ and $\varphi'(x) = 0$ for $x \in \mathcal{O}$.

Proposition 2.21. Let φ be the Cantor-Lebesgue function and define the function ψ on $[0,1]$ by $\psi(x) = \varphi(x) + x$. Then ψ is a strictly increasing continuous function that maps $[0,1]$ onto $[0,2]$,

(i) maps the Cantor set $\mathbf{C}$ onto a measurable set of positive measure and

(ii) maps a measurable set, a subset of the Cantor set, onto a nonmeasurable set.

Proposition 2.22. There is a measurable set, a subset of the Cantor set, that is not a Borel set.

Note. Notice that Proposition 2.22 implies that the σ-algebra of Borel sets, $\mathcal{B}$, do not include all measurable sets (in fact, $\mathcal{B}$ does not even include all measure zero sets). In the supplement to this section of notes, it is argued that $|\mathcal{M}| = |\mathcal{M}_0| = |\mathcal{P}(\mathbb{R})| = \aleph_2$ (where $\mathcal{M}_0$ denotes the set of all measure zero sets), whereas (as claimed in Section 1.4) $|\mathcal{B}| = \aleph_1$.

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