Chapter 20. The Construction of Particular Measures

Note. In this chapter, we return to the "concrete" setting of measure theory, $\mathbb{R}$. But first, we define a measure on the Cartesian product of two (abstract) measure spaces. This approach is used to define a measure on $\mathbb{R}^n$ which in turn produces a Lebesgue integral for measurable functions defined on $\mathbb{R}^n$.

Section 20.1. Product Measures: The Theorems of Fubini and Tonelli

Note. In this section, we use the Carathéodory-Hahn Theorem to define a measure on the Cartesian product of two measure spaces in terms of the measures on the constituent spaces. This section is in the abstract setting, but in the next section we consider $\mathbb{R}^2$ and $\mathbb{R}^n$.

Definition. Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two measure spaces. We consider the Cartesian product $X \times Y$ and for $A \subset X$ and $B \subset Y$, we call $A \times B$ a *rectangle*. If $A \in \mathcal{A}$ and $B \in \mathcal{B}$, we call $A \times B$ a *measurable rectangle* provided $\mu(A) < \infty$ and $\nu(B) < \infty$.

Note. The obvious choice for the measure of $A \times B$ is $\mu(A) \cdot \nu(B)$. In order to show that this works, we first need a lemma concerning decompositions of measurable rectangles into measurable rectangles.

Lemma 20.1. Let $\{A_k \times B_k\}_{k=1}^{\infty}$ be a countable disjoint collection of measurable rectangles whose union also is a measurable rectangle $A \times B$. Notice that index k ranges over ALL of the rectangles which compose $A \times B$ (so there is no $A_i \times B_j$ where $i \neq j$). Then

$$\mu(A) \cdot \nu(B) = \sum_{k=1}^{\infty} \mu(A_k) \cdot \nu(B_k).$$

Note. The following result allows us to use the Carathéodory-Hahn Theorem to produce a measure on $X \times Y$ using measurable rectangles.

Proposition 20.2. Let $\mathcal{R}$ be the collection of measurable rectangles in $X \times Y$ and for a measurable rectangle $A \times B$ define $\lambda(A \times B) = \mu(A) \cdot \nu(B)$. Then $\mathcal{R}$ is a semiring and $\lambda : \mathcal{R} \to [0, \infty]$ is a premeasure.

Definition. Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be measure spaces, $\mathcal{R}$ the collection of measurable rectangles in $X \times Y$, and λ the premeasure defined on $\mathcal{R}$ by $\lambda(A \times B) = \mu(A) \cdot \nu(B)$ for $A \times B \in \mathcal{R}$. The *product measure* $\lambda = \mu \times \nu$ is the Carathéodory extension of $\lambda : \mathcal{R} \to [0, \infty]$ defined on the σ-algebra of $(\mu \times \nu)^*$-measurable subsets of $X \times Y$.

Definition. Let $E \subset X \times Y$ and let f be an extended real valued function on E. For point $x \in X$, the set $E_x = \{y \in Y \mid (x, y) \in E\} \subset Y$ is the *x-section of set E* and the function $f(x, \cdot)$ defined on E_x as $f(x, \cdot)(y) = f(x, y)$ is the *x-section of function f*.

Note. We are interested in computing double integrals using iterated integration (as in Calculus III). The "Theorems of Fubini and Tonelli" are the main results on this.

Fubini's Theorem.

Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be measure spaces and let ν be complete. Let f be integrable over $X \times Y$ with respect to the product measure $\mu \times \nu$. Then for almost all $x \in X$, the x-section of f, $f(x, \cdot)(y)$, is integrable over Y with respect to ν and

$$\int_{X \times Y} f \, d(\mu \times \nu) = \int_X \left[\int_Y f(x, y) \, d\nu(y) \right] d\mu(x).$$

Note. So Fubini's Theorem relates the double integral to the iterated integral first with respect to y and then with respect to x. We need several preliminary results before we can prove Fubini's Theorem.

Lemma 20.3. Let $E \subset X \times Y$ be an $\mathcal{R}_{\sigma\delta}$ set for which $(\mu \times \nu)(E) < \infty$. Then for all $x \in X$, the x-section of set E, E_x, is a ν-measurable subset of Y, the function $x \mapsto \nu(E_x)$ for $x \in X$ is a μ-measurable function and

$$(\mu \times \nu)(E) = \int_X \nu(E_x)\, d\mu(x).$$

Note. The conclusion of the following result is the same as Lemma 20.3, but the setting now involves sets of product measure zero. Of course, we can have $(\mu \times \nu)(E) = 0$ yet have the x-section not be of ν-measure 0 (or even necessarily be ν-measurable). This could occur if the "μ part" of E has measure zero (with a rectangle, for example).

Lemma 20.4. Assume the measure ν is complete. Let $E \subset X \times Y$ be measurable with respect to $\mu \times \nu$. If $(\mu \times \nu)(E) = 0$, then almost all $x \in X$, the x-section of E, E_x, is ν-measurable and $\nu(E_x) = 0$. Therefore

$$0 = (\mu \times \nu)(E) = \int_X \nu(E_x)\, d\mu(x) = 0.$$

Note. The following result is similar to the previous two, but deals with general measurable sets E of finite measure.

Proposition 20.5. Assume the measure ν is complete. Let $E \subset X \times Y$ be measurable with respect to $\mu \times \nu$ and $(\mu \times \nu)(E) < \infty$. Then for almost all $x \in X$, the x-section of E, E_x, is a ν-measurable subset of Y, the function $x \mapsto \nu(E_x)$ is μ-measurable for all $x \in X$, and

$$(\mu \times \nu)(E) = \int_X \nu(E_x)\, d\mu(x).$$

Note. The following gets us much closer to Fubini's Theorem and shows that double integrals of simple functions can be calculated using iterated integrals.

Theorem 20.6. Assume measure ν is complete. Let $\phi : X \times Y \to \mathbb{R}$ be a simple function that is integrable over $X \times Y$ with respect to $\mu \times \nu$. Then for almost all $x \in X$, the x-section of ϕ, $\phi(x, \cdot)$, is integrable over Y with respect to ν and

$$\int_{X \times Y} \phi\, d(\mu \times \nu) = \int_X \left[\int_Y \phi(x, y)\, d\nu(y) \right] d\mu(x).$$

Note. We are now ready to prove Fubini's Theorem.

Note. So Fubini's Theorem allows us to calculate a double integral of an integrable function using iterated integrals. As given, we first integrate with respect to ν and then with respect to μ. The following result (really, the corollary to it) allows us to perform the iterated integration in the opposite order (for nonnegative functions).

Tonelli's Theorem.

Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two σ-finite measure spaces and ν be complete. Let f be a nonnegative $(\mu \times \nu)$-measurable function on $X \times Y$. Then for almost all $x \in X$, the x-section of function f, $f(x, \cdot)$, is ν-measurable and the function defined a.e. on X by

$$x \mapsto (\text{the integral of } f(x, \cdot) \text{ over } X \text{ with respect to } \nu)$$

is μ-measurable. Moreover,

$$\int_{X \times Y} f \, d(\mu \times \nu) = \int_X \left[\int_Y f(x, y) \, d\nu(y) \right] d\mu(x).$$

Note. Both Fubini and Tonelli are stated in terms of integration first with respect to y and then with respect to x. Of course, this can be reversed in both results provided we have the hypothesis that μ is complete.

Corollary 20.7. (Tonelli's Corollary).

Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two σ-finite, complete measure spaces and f a non-negative $(\mu \times \nu)$-measurable function of $X \times Y$. Then:

(i) For almost all $x \in X$, the x-section of f, $f(x, \cdot)$, is ν-measurable and the function defined almost everywhere on X by

$$x \mapsto (\text{the integral of } f(x, \cdot) \text{ over } X \text{ with respect to } \nu)$$

is μ-measurable, and

(ii) for almost all $y \in Y$, the y-section of f, $f(\cdot, y)$, is μ-measurable and the function defined almost everywhere on Y by

$$y \mapsto (\text{the integral of } f(\cdot, y) \text{ over } Y \text{ with respect to } \mu)$$

is μ-measurable.

Moreover, if

$$\int_X \left[\int_Y f(x, y)\, d\nu(y) \right] d\mu(x) < \infty,$$

then f is integrable over $X \times Y$ with respect to $\mu \times \nu$ and

$$\int_Y \left[\int_X f(x, y)\, d\mu(x) \right] d\nu(y) = \int_{X \times Y} f\, d(\mu \times \nu) = \int_X \left[\int_Y f(x, y)\, d\nu(y) \right] d\mu(x).$$

Note. We now have three results concerning iterated integrals: (1) Fubini's Theorem, (2) Tonelli's Theorem, and (3) Tonelli's Corollary (Corollary 20.7). Here's how they compare:

(1) Fubini's Theorem requires integrability of f over $X \times Y$ and completeness of ν. Surprisingly, the iterated integral may be finite, but f not integrable over $X \times Y$ (see Problem 20.6).

(2) Tonelli's Theorem requires nonnegativity of f, the completeness of ν, AND the σ-finiteness of both measure spaces. This result does not hold if we omit either the nonnegativity or the σ-finiteness (see Problem 20.5).

(3) Tonelli's Corollary requires nonnegativity of f, the completeness of both μ and ν, and the σ-finiteness of both measure spaces. These hypotheses guarantee the measurability of the x-sections and y-sections of f. In addition ("Moreover"), the hypothesis of finiteness of one of the iterated integrals (either one, actually) insures the integrability of f over $X \times Y$ and that the integral of f can be calculated using either iterated integral.

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