Section 20.2. Lebesgue Measure on Euclidean Space $\mathbb{R}^n$

Note. In this section, we extend Lebesgue measure to $\mathbb{R}^n$ using the Carathéodory-Hahn Theorem. The Lebesgue integral of real valued functions on subsets of $\mathbb{R}^n$ will then follow by the techniques of Chapter 18 ("Integration Over General Measure Spaces").

Note. Recall that $\mathbb{R}^n\{(x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R}^n \text{ for } 1 \leq i \leq n\}$, $\mathbb{R}^n$ is a linear space with bilinear form $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined as $\langle x, y \rangle = \sum_{k=1}^n x_k y_k$ (the usual inner product on $\mathbb{R}^n$, and the induced norm $\|x\| = \sqrt{\langle x, x \rangle} = \left\{\sum_{k=1}^n x_k^2\right\}^{1/2}$.

Note. Royden and Fitzpatrick deviate from tradition and consider $[a, b]$, $[a, b)$, $(a, b]$, and (a, b) for $a \leq b$ as "bounded intervals." Notice that this implies that singletons are then intervals. We have the length of such bounded inverval I as $\ell(I) = b - a$.

Definition. A *bounded interval* in $\mathbb{R}^n$ is a set I that is the Cartesian product of n bounded intervals of real numbers, $I = I_1 \times I_2 \times \cdots \times I_n$. The *volume* of I is $\text{vol}(I) = \ell(I_1)\ell(I_2)\cdots\ell(I_n)$.

Note. The following is introduced to set up a situation where we can deal with volumes of bounded intervals in $\mathbb{R}^n$ in a way that does not (directly) reference the endpoints of the constituent intervals I_k $(1 \leq k \leq n)$.

Definition. A point in $\mathbb{R}^n$ is an *integral point* if each of its coordinates is an integer. For bounded interval I in $\mathbb{R}^n$, the *integral count*, $\mu^{integral}(I)$, is the number of integral points in I.

Lemma 20.8. For each $\varepsilon > 0$, the ε-*dilation* $T_\varepsilon : \mathbb{R}n \to \mathbb{R}^n$ is $T_\varepsilon(x) = \varepsilon n$. Then for each bounded interval I in $\mathbb{R}^n$,

$$\lim_{\varepsilon \to 0} \frac{\mu^{integral}(T_\varepsilon(I))}{\varepsilon^n} = \text{vol}(I).$$

Note. In Exercise 20.25 it is to be shown that the Cartesian product of two semirings is a semiring and then that the following holds by induction.

Proposition 20.9. The collection $\mathcal{I}$ of bounded intervals in $\mathbb{R}^n$ is a semiring.

Note. We now show that the volume of an interval I in $\mathbb{R}^n$ is a premeasure. Then we can use the results of Section 17.5, "The Carathéodory-Hahn Theorem: The Extension of a Premeasure to a Measure," to define an outer measure and measure on $\mathbb{R}^n$.

Proposition 20.10. The set function volume, $\text{vol} : \mathcal{I} \to [0, \infty)$, is a premeasure on the semiring $\mathcal{I}$ of bounded intervals in $\mathbb{R}^n$.

Note. Recall from Theorem 17.9 that the outer measure induced by set function μ defined on collection of sets $\mathcal{S}$ is defined as $\mu^*(E) = \inf \sum_{k=1}^{\infty} \mu(E_k)$ where the infimum is taken over all countable collections $\{E_k\}_{k=1}^{\infty}$ of sets in $\mathcal{S}$ that cover E. This allows us to use the premeasure vol to define an outer measure on $\mathbb{R}^n$.

Definition. The outer measure μ_n^* induced by premeasure vol on the semiring of bounded intervals in $\mathbb{R}^n$ is the *Lebesgue outer measure* on $\mathbb{R}^n$. The collection of μ_n^*-measurable sets (in the sense of Carathéodory, as defined in Section 17.3, "The Carathéodory Measure Induced by an Outer Measure") is denoted $\mathcal{L}^n$ and called the *Lebesgue measurable* sets in $\mathbb{R}^n$. The restriction of μ_n^* to $\mathcal{L}^n$ is *Lebesgue measure* on $\mathbb{R}^n$ (or "n-dimensional Lebesgue measure") and denoted by μ_n.

Note. We now confirm that $\mathcal{L}^n$ is a σ-algebra and that for a bounded interval I in $\mathbb{R}^n$, $\mu_n(I) = \text{vol}(I)$ (analogous to the fact that for an interval in $\mathbb{R}$, its Lebesgue measure is its length).

Theorem 20.11. The σ-algebra $\mathcal{L}^n$ of Lebesgue measurable subsets of $\mathbb{R}^n$ contains the bounded intervals in $\mathbb{R}^n$ and contains the Borel subsets in $\mathbb{R}^n$. Moreover, the measure space $(\mathbb{R}^n, \mathcal{L}^n, \mu_n)$ is both σ-finite and complete. For bounded interval I in $\mathbb{R}^n$, $\mu_n(I) = \text{vol}(I)$.

Corollary 20.12. Let E be a Lebesgue measurable subset of $\mathbb{R}^n$ and $f : E \to \mathbb{R}$ be continuous. Then f is measurable with respect to n-dimensional Lebesgue measure.

Note. The next result shows that Lebesgue measure for measurable sets is the same as outer measure (defined using open sets) and inner measure (defined using compact sets); see my online handout on "An Alternate Approach to the Measure of a Set of Real Numbers" at: `http://faculty.etsu.edu/gardnerr/talks/Measure-Theory.pdf`.

Theorem 20.13. Let B be a Lebesgue measurable subset of $\mathbb{R}^n$. Then

$$\mu(E) = \inf\{\mu_n(\mathcal{O}) \mid E \subset \mathcal{O}, \mathcal{O} \text{ is open}\}$$

and

$$\mu(E) = \sup\{\mu_n(\mathcal{K}) \mid \mathcal{K} \subset E, \mathcal{K} \text{ is compact}\}.$$

Note. Recall that in Theorem 2.11 it was shown that the measurability of E is equivalent to the existence of G_δ set G with $E \subset G$ and $m^*(G \setminus E) = 0$ (G is the outer approximation of E with a G_δ set) and equivalent to the existence of F_σ set F with $F \subset E$ and $m^*(E \setminus F) = 0$ F is the inner approximation of E with an F_σ set). A similar result for a Carathédory measure is given in Proportion 17.10. Since $\mathcal{L}^n$ contains the Borel sets by Theorem 20.11, then $\mathcal{L}^n$ contains the G_δ and F_σ sets. So, as a corollary to Theorem 20.13 we have the following result analogous to Theorem 2.11.

Corollary 20.14. For a subset E of $\mathbb{R}^n$, the following assertions are equivalent:

(i) E is measurable with respect to n-dimensional Lebesgue measure.

(ii) There is a G_δ subset G of $\mathbb{R}^n$ such that $E \subset G$ and $\mu_n^*(G \setminus E) = 0$.

(iii) There is a F_σ subset F of $\mathbb{R}^n$ such that $F \subset E$ and $\mu_n^*(E \setminus F) = 0$.

Note/Definition. In Problem 20.20, it is to be shown that Lebesgue measure is translation invariant in the following sense. For $E \subset \mathbb{R}^n$ and $z \in \mathbb{R}^n$, define the translation $E + Z = \{x + z \mid z \in E\}$. Then μ_n is *translation invariant* if for E μ_n-measurable we have $E + z$ is μ_n-measurable and $\mu_n(E) = \mu_n(E + z)$.

Note. In the previous section, we took measure spaces $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \eta)$ and defined a premeasure on the measurable rectangles in $X \times Y$ as $\lambda(A \times B) = \mu(A) \cdot \eta(B)$. The product measure was then defined as the Carathéodory extension of λ. We now show that Lebesgue measure on $\mathbb{R}^n$ can be expressed as a product of two measures, one on $\mathbb{R}^m$ and the other on $\mathbb{R}^k$ where $n = m + k$.

Definition. Consider the sets $\mathbb{R}^n$, $\mathbb{R}^m$, $\mathbb{R}^k$, and $\mathbb{R}^m \times \mathbb{R}^k$ where $n, m, k \in \mathbb{Q}$ and $n = m + k$. Define the mapping $\varphi : \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^k$ as

$$\varphi((x_1, x_2, \ldots, x_n)) = ((x_1, x_2, \ldots, x_m), (x_{m+1}, x_{m+2}, \ldots, x_{m+k})) \in \mathbb{R}^m \times \mathbb{R}^k \qquad (20)$$

for $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.

Note. Mapping φ is one to one and onto. Each of $\mathbb{R}^n$, $\mathbb{R}^m$, and $\mathbb{R}^k$ has a linear structure (as a vector space), a topological structure, and a measure structure (the Lebesgue measures μ_n, μ_m, and μ_k, respectively). The product space $\mathbb{R}^m \times \mathbb{R}^k$ inherits a linear structure, a topological structure (using the product topology), and a measure structure from its component spaces $\mathbb{R}^m$ and $\mathbb{R}^k$. The mapping φ is an isomorphism with respect to the linear structure and the topological structure (since the projection mappings are continuous under the product topology by Proposition 12.4). The next result shows that φ is also an isomorphism from measure space $(\mathbb{R}^n, \mathcal{L}^n, \mu_n)$ to $(\mathbb{R}^m \times \mathbb{R}^k, \mathcal{L}, \mu_m \times \mu_n)$.

Proposition 20.15. For the mapping $\varphi : \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^k$ defined by (20), a subset E of $\mathbb{R}^n$ is measurable with respect to n-dimensional Lebesgue measure μ_n if and only if its image $\varphi(E)$ is measurable with respect to the product measure $\mu_m \times \mu_k$ on $\mathbb{R}^m \times \mathbb{R}^k$ and $\mu_n(E) = (\mu_m \times \mu_k)(\varphi(E))$.

Note. Proposition 20.15, combined with the Theorems of Fubini and Tonelli, allow us to easily prove the following (as is asked in Exercise 20.2.E).

Theorem 20.16. For $n, m, k \in \mathbb{N}$ such that $n = m + k$, consider the mapping $\varphi : \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^k$ defined by (20). A function $f : \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}$ is measurable with respect to the product measure $\mu_m \times \mu_k$ if and only if the composition $f \circ \varphi : \mathbb{R}^n \to \mathbb{R}$ is measurable with respect to Lebesgue measure μ_n. If f is integrable over $\mathbb{R}^n$ with respect to Lebesgue measure μ_n then

$$\int_{\mathbb{R}^n} f \, d\mu_n = \int_{\mathbb{R}^k} \left(\int_{\mathbb{R}^m} f(x, y) \, d\mu_m(x) \right) d\mu_k(y).$$

Moreover, if f is nonnegative and measurable with respect to Lebesgue measure μ_n, this equality also holds.

Note. Let $\mathcal{L}(\mathbb{R}^n)$ be the linear space of linear operators $T : \mathbb{R}^n \to \mathbb{R}^n$. Denote by $GL(n, \mathbb{R})$ the subset of $\mathcal{L}(\mathbb{R}^n)$ of one to one and onto (and hence invertible) operators. Under the binary operation of composition, $GL(n, \mathbb{R})$ is a group, the *general linear group*. In fact, $GL(n, \mathbb{R})$ is isomorphic to the group of all invertible $n \times n$ matrices with real entries. See my online notes for Introduction to Modern Algebra (MATH 4127/5127), Section I.4, "Groups": `http://faculty.etsu.edu/gardnerr/4127/notes/I-4.pdf`.

Note. Recall that $f : X \to Y$, where (X, ρ) and (Y, σ) are metric spaces, is *Lipschitz* if there is $c \geq 0$ such that for all $u, v \in X$ we have $\sigma(f(u), f(v)) \leq c\rho(u, v)$ (see Section 9.3). A Lipschitz function on X is uniformly continuous on X (let $\delta = \varepsilon/c$).

Proposition 20.17. A linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz.

Note. In Proposition 2.20(ii), it is shown that there is a continuous function ψ (related tot the Cantor-Lebesgue function $\varphi(x)$ as $\psi(x) = \varphi(x) + x$) which maps a measurable set of real numbers onto a set of nonmeasurable set of real numbers. So continuous functions do not preserve the property of measurability. In Exercise 2.38 it is to be shown that a Lipschitz function maps measurable sets of real numbers to measurable sets of real numbers. The second result also holds for Lipschitz linear mappings.

Proposition 20.18. Let the mapping $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ be Lipschitz. If E is a Lebesgue measurable subset in $\mathbb{R}^n$, so is $\Psi(E)$. In particular, a linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ maps Lebesgue measurable sets to Lebesgue measurable sets.

Note. It shouldn't be surprising that Proposition 20.18 implies that the composition of a linear operator with a measurable function is a measurable function, as now given.

Corollary 20.19. Let the function $f : \mathbb{R}^n \to \mathbb{R}$ be measurable with respect to Lebesgue measure and let the operator $T : \mathbb{R}^n \to \mathbb{R}^n$ be linear and invertible. Then the composition $f \circ T : \mathbb{R}^n \to \mathbb{R}$ is also measurable with respect to Lebesgue measure.

Note. You may have seen in sophomore Linear Algebra (MATH 2010) that if B is an n-box in $\mathbb{R}^n$ of volume V and $A : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation (so A is an $n \times n$ matrix) then the volume of the image of B under transformation A, AB, is $|\det(A)|V$. Since integrals (and volumes) in $\mathbb{R}^n$ are defined, as here, in terms of n-boxes (or "intervals" here) then the volume V of any measurable region in $\mathbb{R}^n$ maps to a region of volume $|\det(A)|V$ under linear transformation A. The factor $|\det(A)|$ is the *volume-change factor* of linear transformation A. See my online notes on "Linear Transformations and Determinants" at: `http://faculty.etsu.edu/gardnerr/2010/c4s4.pdf`. We now show this idea holds for Lebesgue integrals when $n = 1$ (in which case the linear transformation maps $x \mapsto \alpha x$ for some $\alpha \in \mathbb{R}$).

Proposition 20.20. Let $f : \mathbb{R} \to \mathbb{R}$ be integrable over $\mathbb{R}$ with respect to one-dimensional Lebesgue measure μ_1. If $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0$, then

$$\int_{\mathbb{R}} f\, d\mu_1 = |\alpha| \int_{\mathbb{R}} f(\alpha x)\, d\mu_1(x) \text{ and } \int_{\mathbb{R}} f\, d\mu_1 = \int_{\mathbb{R}} f(x+\beta)\, d\mu_1(x).$$

Note. In the next result we consider linear transformations mapping $\mathbb{R}^2 \to \mathbb{R}^2$. We consider three particular transformations,

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, A_3 = \begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}.$$

A_1 corresponds to reflection about the line $y = x$ in the xy-plane, A_2 corresponds to a (particular) vertical shear in the xy-plane, and A_3 corresponds to horizontal expansion.contraction (for $c > 0$). With $c = -1$ in A_3, $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ is a reflection about the y-axis. With $c = -1$ in A_3, we have $A_1A_3A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is a reflection about the x-axis. Now $A_1A_3A_1 = \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}$ is a vertical expansion/contraction (for $c > 0$). Let $A_4 = \begin{bmatrix} 1/c & 0 \\ 0 & 1 \end{bmatrix}$ (so that A_4 is of the *form* of A_3), then $A_4A_2A_3 = \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$ is a horizontal shear. Finally, $A_3A_1A_2A_1A_4 = \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$ which is a vertical shear. Now every invertible linear transformation mapping $\mathbb{R}^2 \to \mathbb{R}^2$ is a finite sequence of

- reflections about the x-axis, y-axis, or the line $y = x$,
- vertical or horizontal expansions or contractions, and
- vertical or horizontal shears.

See Theorem 2.4.A in my online notes for "Geometric Description of Invertible Transformations of $\mathbb{R}^2$," at `http://faculty.etsu.edu/gardnerr/2010/c2s4.pdf`. So by considering transformations of the forms A_1, A_2, and A_3, we have through compositions all invertible transformations of $\mathbb{R}^2$.

Proposition 20.21. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be integrable over $\mathbb{R}^2$ with respect to Lebesgue measure μ_2 and let $c \neq 0$ be a real number. Define $\varphi : \mathbb{R}^2 \to \mathbb{R}$, $\psi : \mathbb{R}^2 \to \mathbb{R}$, and $\eta : \mathbb{R}^2 \to \mathbb{R}$ by $\varphi(x,y) = f(y,x)$, $\psi(x,y) = f(x, x+y)$, and $\eta(x,y) = f(cx,y)$ for all $(x,y) \in \mathbb{R}^2$. Then φ, ψ, and η are integrable over $\mathbb{R}^2$ with respect to Lebesgue measure μ_2. Moreover,

$$\int_{\mathbb{R}^2} f\, d\mu_2 = \int_{\mathbb{R}^2} \varphi\, d\mu_2 = \int_{\mathbb{R}^2} \psi\, d\mu_2 \text{ and } \int_{\mathbb{R}^2} f\, d\mu_2 = |c| \int_{\mathbb{R}^2} \eta\, d\mu_2.$$

Note. We now five a proof of Proposition 20.21 for φ and leave the proofs for ψ and η as Exercise 20.2.F.

Note. We need a couple of results from matrix theory. We know that every element of $GL(n,\mathbb{R})$ can be represented by an $n \times n$ invertible matrix. Such a matrix is row equivalent to the $n \times n$ identity matrix $\mathcal{I}$ (see Theorem 1.12 of my online Linear Algebra notes at `http://faculty.etsu.edu/gardnerr/2010/c1s5.pdf`. Recall that the elementary row operations on a matrix are:

(1) multiplying the jth row by a nonzero scalar c,

(2) interchanging two rows,

(3) adding a multiple of one row to another.

Royden and Fitzpatrick use this to motivate consideration of three types of elementary linear operators. They describe them as they affect the standard basis vectors $e_1, e_2, \ldots, e_n$ (which we might view as how the transformation affects the rows of an $n \times n$ identity matrix:

Type 1. $T(e_j) = ce_j$ and $T(e_k) = e_k$ for $k \neq j$;

Type 2. $T(e_j) = e_{j+1}$, $T(e_{j+1}) = e_j$, and $T(e_k) = e_k$ for $k \notin \{j, j+1\}$;

Type 3. $T(e_j) = e_j + e_{j+1}$ and $T(e_k) = e_k$ for $k \neq j$.

The Type 2 linear operators interchange consecutive basis vector, but with a sequence of these operators any two basis vectors can be interchanged. Type 3 linear operators add one basis element to another (name, maps e_j to $e_j + e_{j+1}$), and this combined with a Type 1 linear operator and a sequence of Type 2 linear operators allow us to add a multiple of any basis vector to any other. So all elementary row operations are produced from this list of three "very elementary" linear operators.

Note. The other idea we need from matrix theory is the idea of a determinant of a matrix (or for a "linear transformation" here). We denote the determinant of linear transformation T as $\det(T)$ and Royden and Fitzpatrick note that $\det(T)$ satisfies:

(i) For any two linear operators $T, S : \mathbb{R}^n \to \mathbb{R}^n$ we have $\det(S \circ T) = \det(S)\det(T)$;

(ii) If T is Type 1 then $\det(T) = c$, if T is Type 2 then $\det(T) = -1$, and if T is Type 3 then $\det(T) = \det(\mathcal{I}) = 1$;

(iii) If $T(e_n) = e_n$ and T maps the subspace $\{x \in \mathbb{R}^n \mid x = (x_1, x_2, \ldots, x_{n-1}, 0)\}$ into itself then $\det(T) = \det(T')$ where $T' : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ is the restriction of T to $\mathbb{R}^{n-1}$.

These properties are seen in Linear Algebra (MATH 2010). See my online notes on "The Determinant of a Square Matrix" at `http://faculty.etsu.edu/gardnerr/`

2010/c4s2.pdf. Property (i) is Theorem 4.4 ("The Multiplicative Property") in the Linear Algebra notes, Properties (ii) are implied by Theorem 4.2.A ("Properties of the Determinant") in the Linear Algebra notes, and Property (iii) is implied by the recursive definition of determinant (see Definition 4.1b in the Linear Algebra notes).

Note. In Proposition 20.20, we considered $(f \circ T)(x) = f(\alpha x)$ so that we had the linear transformation $T(x) = \alpha x$ which can be viewed as action by a 1×1 matrix $[\alpha]$ which has determinant α, $\det(T) = \alpha$. In Proposition 20.21, we considered $\eta(x, y) = f(cx, y)$ which we can view as $(f \circ T)(x, y)$ where $T : \mathbb{R}^2 \to \mathbb{R}^n$ is represented by the matrix $= \begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}$ and so $\det(T) = c$. So in Propositions 20.20 and 20.21, we see that integrals of compositions of f with a linear transformation mapping $\mathbb{R}^n \to \mathbb{R}^n$ (where $n \in \{1, 2\}$) are related to the original integral of f by a factor of $|\det(T)|$. The next result shows that this holds for all $n \in \mathbb{N}$.

Proposition 20.22. Let the linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ be invertible and the function $f : \mathbb{R}^n \to \mathbb{R}$ be integrable over $\mathbb{R}^n$ with respect to Lebesgue measure μ_n. Then the composition $f \circ T : \mathbb{R}^n \to \mathbb{R}$ is also integrable over $\mathbb{R}^n$ with respect to Lebesgue measure μ_n and

$$\int_{\mathbb{R}^n} f \, d\mu_n = |\det(T)| \int_{\mathbb{R}^n} f \circ T \, d\mu_n \text{ or } \int_{\mathbb{R}^n} f \circ T \, d\mu_n = \frac{1}{|\det(T)|} \int_{\mathbb{R}^n} f \, d\mu_n.$$

Note. You may have seen a generalization of this in Calculus 2 (MATH 2110) when using substitution in multiple integrals. Recall that in a double integral (under certain hypotheses) if we make a substitution $x = g(u, v)$ and $y = h(u, v)$ then

$$\int\int_R f(x, y)\, dx\, dy = \int\int_G f(g(u, v), h(u, v))|J(u, v)|\, du\, dv$$

where G is the image of R under the substitution mapping and

$$J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

is the *Jacobian determinant* (or simply "Jacobian") of the substitution. Notice that if the substitution is linear (that is, g and h are linear functions of u an dv) then the Jacobian is just the determinant of a 2×2 matrix of constants and this result reduces to Theorem 20.11. My online Calculus 3 notes also state a similar result for $f(x, y, z)$; see "Substitutions in Multiple Integrals" at `http://faculty.etsu.edu/gardnerr/2110/notes-12e/c15s8.pdf`. This can be extended to functions of n-variables.

Note. The next result shows that invertible linear operators preserve measurable sets and affect Lebesgue measure in a predictable way.

Corollary 20.23. Let the linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ be invertible. Then for each Lebesgue measurable subset E of $\mathbb{R}^n$, $T(E)$ is Lebesgue measurable and $\mu_n(T(E)) = |\det(T)|\mu_n(E)$.

Definition. A *rigid motion* of $\mathbb{R}^n$ is a mapping ψ of $\mathbb{R}^n \to \mathbb{R}^n$ that preserves Euclidean distances between points; that is

$$\|\psi(u) - \psi(v)\| = \|u - v\| \text{ for all } u, v \in \mathbb{R}^n.$$

Note. In the setting of metric spaces, a rigid motion is called an "isometry" (see Section 9.1, "Examples of Metric Spaces"). Royden and Fitzpatrick use a theorem of Mazur and Ulam (see page 434) to argue that rigid motions preserve Lebesgue measure, as follows.

Corollary 20.24. Let $\psi : \mathbb{R}^n \to \mathbb{R}^n$ be a rigid motion. Then for each Lebesgue measurable subset E of $\mathbb{R}^n$, $\mu_n(\psi(E)) = \mu_n(E)$.

Note. If a linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ fails to be invertible then its range is some subspace of $\mathbb{R}^n$ of dimension less than n (the rank of the matrix associated with T must be less than n [see Theorem 2.6 of my notes on "The Rank of a Matrix" at `http://faculty.etsu.edu/gardnerr/2010/c2s2.pdf`] and the range of T is the column space of the associated matrix). So the μ_n measure of $T(E)$ for some measurable $E \subset \mathbb{R}^m$ is then 0. Also, $\det(T) = 0$ in this case so Corollary 20.23 in fact holds for noninvertible linear operators also. Here, Royden and Fitzpatrick are only addressing the μ_n-measure of subsets of $\mathbb{R}^n$. If a set E lies in some subspace of $\mathbb{R}^n$ of dimension m, then we could address the μ_m-measure of set E. Fraleigh and Beauregard in *Linear Algebra*, 3rd Edition (Addison-Wesley, 1995) give a fairly nice version of discussing this in terms of n-boxes and m-boxes and transformations mapping $\mathbb{R}^n \to \mathbb{R}^m$ is their Section 4.4, "Linear Transformations and Determinants." So also my online notes at `http://faculty.etsu.edu/gardnerr/2010/c4s4.pdf`.

Revised: 12/20/2018