Section 20.3. Cumulative Distribution Functions and Borel Measures on $\mathbb{R}$

Note. In this section, we return to a study of intervals and Borel subsets of an interval. We define a Borel measure and a cumulative distribution function of a Borel measure. We relate the two using an increasing "continuous on the right" function. We also introduce the Lebesgue-Stieltjes integral.

Recall. The Borel sets on $\mathbb{R}$ are the smallest σ-algebra containing the open sets. A function g defined on interval $[c, d)$ is *continuous on the right* if $\lim_{x \to c^+} g(x) = g(c)$.

Definition. Let $I = [a, b]$ be a closed, bounded interval of real numbers and $\mathcal{B}(I)$ the collection of Borel subsets of I. A finite measure μ on $\mathcal{B}(I)$ is a *Borel measure*. For such a measure, define the function $g_\mu : I \to \mathbb{R}$ by $g_\mu(x) = \mu([a, x])$ for all $x \in I$. The function g_μ is the *cumulative distribution function* of μ.

Note. The following result relates cumulative distribution functions and increasing functions continuous from the right.

Proposition 20.25. Let μ be a Borel measure on $\mathcal{B}(I)$. Then its cumulative distribution function g_μ is increasing and continuous on the right. Conversely, each function $g : I \to \mathbb{R}$ is increasing and continuous on the right is the cumulative distribution function of a unique Borel measure μ_g on $\mathcal{B}(I)$.

Recall. A measure ν on measurable space $(X, \mathcal{M})$ is *absolutely continuous* with respect to measure μ on $(X, \mathcal{M})$, denoted $\nu \ll \mu$, if $E \in \mathcal{M}$ and $\mu(E) = 0$ implies $\nu(E) = 0$. A real valued function f on $[a, b]$ is *absolutely continuous* on $[a, b]$ if for each $\epsilon > 0$, there is a $\delta > 0$ such that for every finite disjoint collection $\{(a_k, b_k)\}_{k=1}^n$ of open intervals in (a, b),

$$\text{if } \sum_{k=1}^n [b_k - a_k] < \delta \text{ then } \sum_{k=1}^n |f(b_k) - f(a_k)| < \epsilon.$$

The following result relates the continuity of a Borel measure to the continuity of its cumulative distribution function. It's proof is to be given in Exercise 20.35

Proposition 20.26. Let μ be a Borel measure on $\mathcal{B}(I)$ and g_μ its cumulative distribution function. Then the measure μ is absolutely continuous with respect to Lebesgue measure if and only if the function g_μ is absolutely continuous.

Note. We now define the Lebesgue-Stieltjes integral and recall the definition of the Riemann-Stieltjes integral from senior level analysis.

Definition. Let $I = [a, b]$ and $g : I \to \mathbb{R}$ be increasing and continuous on the right. Let $f : I \to \mathbb{R}$ be bounded and Borel measurable. The *Lebesgue-Stieltjes integral* of f with respect to g over $[a, b]$ is the integral of f over $[a, b]$ with respect to the Borel measure μ_g, denoted $\int_a^b f\,dg$. That is,

$$\int_z^b f\,dg = \int_{[a,b]} f\,d\mu_g.$$

Note. We now relate Lebesgue-Stieltjes integrals to regular Lebesgue integrals.

Lemma 20.A. If a Borel measure μ is absolutely continuous with respect to Lebesgue measure m, then the Radon-Nikodym derivative of μ with respect to m is the derivative of the cumulative distribution function of μ:

$$\frac{d\mu}{dm} = \frac{d}{dx}[g_\mu(x)] = \frac{d}{dx}[\mu([a,x]).$$

Proof. Exercise 20.44.

Theorem 20.A. Suppose f is a bounded Borel measurable function on $[a,b]$ and g is increasing and absolutely continuous on $[a,b]$. Then for Lebesgue measure m, we have

$$\int_{[a,b]} f\,dg = \int_{[a,b]} fg'\,dm.$$

Proof. Exercise 20.36.

Corollary 20.A. Suppose f is a bounded Borel measurable function on $[a,b]$. Let μ be the Borel measure on $[a,b]$, g_μ the cumulative distribution of μ_g, and m Lebesgue measure. Then

$$\int_{[a,b]} f\,dg_\mu = \int_{[a,b]} f\,\frac{d\mu}{dm}dm$$

where $d\mu/dm$ is the Radon-Nikodym derivative of μ with respect to m.

Proof. By Proposition 20.26, μ_g is absolutely continuous with respect to m. By Lemma, $g'_\mu = \frac{d\mu}{dm}$. The claim follows from Theorem. ∎

Note. Let's recall the definition of the Riemann-Stieltjes integral of f on $[a,b]$ with respect to g.

Definition. Let $P = \{a_0, a_1, \ldots, x_n\}$ be a partition of $[a,b]$ where $a = x_0 < x_1 < \cdots < x_n = b$. Let $\Delta x_i = x_i - x_{i-1}$ and $\|P\| = \max\{\Delta x_i\}$. Let $C = \{c_1, c_2, \ldots, c_n\}$ such that $c_i \in [x_{i-1}, x_i]$. For bounded functions $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$, define the *Riemann-Stieltjes sum*

$$S(f,g,P,C) = \sum_{k=1}^{n} f(c_i)(g(x_i) - g(x_{i-1})).$$

If there is $A \in \mathbb{R}$ such that for each $\epsilon > 0$ there exists $\delta > 0$ where

$$\|P\| < \delta \Rightarrow |S(f,g,P,C) - A| < \epsilon$$

then f is *Riemann-Stieltjes integrable* over $[a,b]$ with respect to g and the *Riemann-Stieltjes integral* is

$$A = \int_a^b f(x)\, dg(x).$$

Note. If $g(x) = x$ then $g(x_i) - g(x_{i-1}) = \Delta x_i$ and Riemann-Stieltjes integration reduces to regular Riemann integration. if g' exists on $[a,b]$, then

$$\int_a^b f(x)\, dg(x) = \int_z^b f(x) g'(x)\, dx,$$

as is shown in senior level analysis.

Note. The following is a relationship between Riemann-Stieltjes integration and Lebesgue-Stieltjes integration.

Theorem 20.B. Suppose f is continuous on $[a, b]$ and g is increasing and absolutely continuous on $[a, b]$. If m is Lebesgue measure, then

$$\int_a^b f(x)\,dg(x) = \int_{[a,b]} fg'\,dm.$$

Proof. Exercise 20.37.

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