Section 20.4. Carathédory Outer Measures and Hausdorff Measures on a Metric Space

Note. Lebesgue outer measure on Euclidean space $\mathbb{R}^n$ has the property that if A and B are subsets of $\mathbb{R}^n$ and there is $\delta > 0$ for which $\|u - v\| \geq \delta$ for all $u \in A$ and $v \in B$, then outer measure μ_n^* satisfies $\mu_n^*(A \cup B) = \mu_n^*(A) + \mu_n^*(B)$. A proof of this result is not in Royden and Fitzpatrick, except that in the case $n = 1$ it appears as Exercise 2.10. In this section, we consider measures induced by outer measures on a metric space that possesses this additivity property.

Definition. Two subsets of set X are *separated* by real-valued function f on X provided there are real numbers a and b with $a < b$ and $f \leq a$ on A and $f \geq b$ on B.

Note. We now show that if when φ separates sets, it implies additivity of the outer measure of those sets, then φ is measurable.

Proposition 20.27. Let φ be a real-valued function on a set X and $\mu^* : 2^X \to [0, \infty]$ an outer measure with the property that whenever two subsets A and B of X are separated by φ, then

$$\mu^*(A \cup B) = \mu^*(A) + \mu^*(B).$$

Then φ is measurable with respect to the measure induced by μ^*.

Note. In a metric space (X, ρ), the distance between sets A and B is defined as $\rho(A, B) = \inf_{u \in A, v \in B} \rho(u, v)$ (see Exercise 9.72). The Borel σ-algebra associated with this metric space, denoted $\mathcal{B}(X)$, is the smallest σ-algebra containing the topology (that is, all open sets under the metric topology).

Definition. Let (X, ρ) be a metric space. An outer measure $\mu^* : 2^X \to [0, \infty]$ is a *Carathéodory outer measure* if for all subsets A and B of X with $\rho(A, B) > 0$, we have

$$\mu^*(A \cup B) = \mu^*(A) + \mu^*(B).$$

Note. We first give a property of a Carathéodory out measure on a metric space and then give an example of a Carathéodory outer measure on any given metric space.

Theorem 20.28. Let μ^* be a Carathéodory outer measure on matrix space (X, ρ). Then every Borel subset of X is measurable with respect to μ^*.

Note. In a metric space (X, ρ), set $A \subset X$ has *diameter* $\operatorname{diam}(A) = \sup_{u,v \in A} \rho(u, v)$ (see Section 9.4).

Definition. Fix $\alpha > 0$. Take $\varepsilon > 0$ and for $E \subset X$ define

$$H_\alpha^{(\varepsilon)} = \inf \sum_{k=1}^{\infty} (\mathrm{diam}(A_k))^\alpha,$$

where $\{A_k\}_{k=1}^{\infty}$ is a countable collection of subsets of X that covers E and each A_k has a diameter less than ε. Define

$$H_\alpha^*(E) = \sup_{\varepsilon > 0} H_\alpha^{(\varepsilon)}(E) = \lim_{\varepsilon \to 0} H_\alpha^{(\varepsilon)}(E).$$

Note. Royden and Fitzpatrick claim that $H_\alpha^{(\varepsilon)}(E)$ increases as ε decreases; thus the expression of $H_\alpha^*(E)$ as a limit is justified. We now show that H_α^* is a Carathéodory out measure on any given metric space (X, ρ).

Proposition 20.29. Let (X, ρ) be a metric space and α a positive real number. Then $H_\alpha^* : 2^X \to [0, \infty]$ is a Carathéodory outer measure.

Note. Since H_α^* is a Carathéodory outer measure then, by Theorem 20.28, it induces a measure on the σ-algebra of Borel sets of X on metric space (X, ρ).

Definition. Let (X, ρ) be a metric space. The measure H_α on the σ-algebra of Borel sets on X, $\mathcal{B}(X)$, is the *Hausdorff α-dimensional measure* on X.

Proposition 20.30. Let (X, ρ) be a metric space. Let A be a Borel subset of X, and let α, β be positive real numbers for which $\alpha < \beta$. If $H_\alpha(A) < \infty$ then $H_\beta(A) = 0$.

Theorem 20.4.A. The Hausdorff 1-dimensional measure, H_1, is the same as Lebesgue measure on the σ-algebra of Lebesgue measurable sets of real numbers.

Note. We might expect Theorem 20.4.A to hold for Hausdorff n-dimensional measure and Lebesgue measure μ_n for all $n \in \mathbb{N}$ (that is, we might expect that $H_n = \mu_n$ on $\mathbb{R}^n$). However, this is not the case for $n > 1$. In Exercise 20.48 it is to be shown that for any bounded set in $\mathbb{R}^2$ that $H_2(A) \geq \frac{4}{\pi}\mu_2(A) > \mu_2(A)$. However, Exercise 20.55 shows that $H_n = \gamma_n \mu_n$ on $\mathbb{R}^n$ for a constant γ_n (in fact, $\gamma_n = H_n(J)$ where J is a unit cube in $\mathbb{R}^n$).

Definition. For $E \subset \mathbb{R}^n$, the *Hausdorff dimension* of E is

$$\dim_{\mathrm{H}}(E) = \inf\{\beta \geq 0 \mid H_\beta(E) = 0\}.$$

Note. Let $E \subset \mathbb{R}^n$ have positive Lebesgue measure, $\mu_n(E)$ (that is, $0 < \mu_n(E) < \infty$). Then by Exercise 20.55, $\infty > H_n(E) = \gamma_n \mu_n(E) > 0$ and so by Proposition 20.30, the Hausdorff dimension is $\dim_{\mathrm{H}}(E) = \inf\{\beta \geq 0 \mid H_\beta(E) = 0\} = n$.

Note. Royden and Fitzpatrick state (page 444): "There are many specific calculations of Hausdorff dimension of subsets of Euclidean space. For instance, it can be shown that the Hausdorff dimension of the Cantor set is $\log 2/\log 3$."

Revised: 12/21/2018