Chapter 22. Invariant Measures

Note. In this chapter, we define topological groups, compact groups, and probability functional. The Kakutani Fixed Point Theorem (Theorem 22.6) guarantees the existence of a particular type of probability functional related to a compact group (Section 22.2). We define a probability measure and Haar measure and prove von Neumann's Theorem (Theorem 22.9) which gives the existence of a Haar related to a compact group (Section 22.3). We define an ergodic transformation. The Bogoliubov-Krilov Theorem (Theorem 22.12) guarantees the existence of a probability measure of a particular type ("measure preserving" with respect to a continuous mapping of a compact metric space to itself).

Note. The necessary background for Section 22.1 includes knowledge of topological spaces (in particular, the Hausdorff topology), Banach spaces, and continuous linear operators on a Banach space. The proof of The Kakutani Fixed Point Theorem requires Alaoglu's Theorem involving normed linear spaces and the weak-$*$ topology (from Section 15.1). the proof of von Neumann's Theorem requires the Kakutani Fixed Point Theorem and Fubini's Theorem (Section 20.1). The proof of the Bogoliubov-Krilov Theorem is based on Helley's Theorem (Section 8.3)

Section 22.1. Topological Groups: The General Linear Groups

Note. In this section we define topological groups, compact groups, and the general linear group of a Banach space. The main result of this section shows that the general linear group of a Banach space is in fact a topological group. A more elementary approach to topological groups can be found in the supplement to Section 22, Quotient Groups, of Munkres *Topology*, 2nd Edition; see my online notes at: `http://faculty.etsu.edu/gardnerr/5357/notes/Munkres-22S.pdf`.

Definition. Let $\mathcal{G}$ be a set of elements with a Hausdorff topology on $\mathcal{G}$ and suppose the elements of $\mathcal{G}$ form a group under binary operation $\cdot$. For $g \in \mathcal{G}$ let g^{-1} denote the inverse of g with respect to $\cdot$ and let e denote the identity element of $\mathcal{G}$ with respect to $\cdot$. Then $\mathcal{G}$ is a *topological group* if

(i) the mapping from $\mathcal{G} \times \mathcal{G}$ to $\mathcal{G}$ defined as $(g_1, g_2) \mapsto g_1 \cdot g_2$ is continuous, where the topology on $\mathcal{G} \times \mathcal{G}$ is the product topology, and

(ii) the mapping from $\mathcal{G}$ to $\mathcal{G}$ defined by $g \mapsto g^{-1}$ is continuous.

Note. As Royden and Fitzpatrick say in the introduction to this chapter, a topological group is a group with a Hausdorff topology such that the group operation and inversion are continuous.

Note. Every group G is a topological group. We just equip G with the discrete topology. Continuity of the binary operation follows at $(x, y) \in G \times G$ by taking the open set $\mathcal{O} = \{x \cdot y\}$ (the "δ set") for any given open set in $G \times G$ containing (x, y) (the "ε set"). In the online notes for Munkres' book *Topology* mentioned above, it is shown that $\mathbb{R}$ under addition, $\mathbb{R}^+$ under multiplication, and $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ under multiplication are each topological groups.

Definition. A topological group which is compact as a topological space is a *compact group.*

Note. Notationally, we have for $\mathcal{G}_1, \mathcal{G}_2 \subset \mathcal{G}$: $\mathcal{G}_1 \cdot \mathcal{G}_2 = \{g_1 \cdot g_2 \mid g_1 \in \mathcal{G}_1, g_2 \in \mathcal{G}_2\}$ and $\mathcal{G}_1^{-1} = \{g^{-1} \mid g \in \mathcal{G}\}$. We also denote $\{g\} \cdot \mathcal{G}_2 = g \cdot \mathcal{G}_2$.

Note. For E a Banach space, $\mathcal{L}(E)$ is the collection of all continuous operators from E to E. By Theorem 13.1, $\mathcal{L}(E)$ consists precisely of the bounded linear operators from E to E and by Theorem 13.3 $\mathcal{L}(E)$ is itself a Banach space. For $S, T \in \mathcal{L}(E)$ and $\alpha, \beta \in \mathbb{R}$ we have for all $f_1, f_2 \in E$ that

$$(S \circ T)(\alpha f_1 + \beta f_2) = S(T(\alpha f_1 + \beta f_2) = S(\alpha T(f_1) + \beta T(f_2))$$

$$= \alpha S(T(f_1)) + \beta S(T(f_2)) = \alpha (S \circ T)(f_1) + \beta (S \circ T)(f_2)$$

and

$$\|(S \circ T)(f)\| = \|S(T(f))\| \leq \|S\|\|T(f)\| \leq \|S\|\|T\|\|f\|$$

so that $\|S \circ T\| \leq \|S\|\|T\|$. so the composition of two operators in $\mathcal{L}(E)$ is itself in $\mathcal{L}(E)$.

Note. Operator $T : E \to E$ is invertible if and only if it is one to one and onto. By the Open Mapping Theorem (actually, Corollary 13.9 following it in Section 13.4), any T^{-1} is continuous. Recall that function composition is associative. So the set of all invertible operators in $\mathcal{L}(E)$ form a group under operator composition.

Definition. Let E be a Banach space and denote the set of invertible operators (that is, one to one and onto operators) in $\mathcal{L}(E)$ as $GL(E)$. With operator composition as the binary operation, $GL(E)$ is the *general linear group* of E. The identity operator is denoted Id.

Note. The general linear group of E is a normed space with the operator norm. The norm induces a metric (where $d(S,T) = \|S - t\|$) and the metric induces a topology on $GL(E)$. This is simply called the topology induced by the norm.

Lemma 22.1. Let E be a Banach space and the operator $C \in \mathcal{L}(E)$ have $\|C\| < 1$. Then $\mathrm{Id} - c$ is invertible and $\|(\mathrm{Id} - C)^{-1}\| \leq (1 - \|C\|)^{-1}$.

Theorem 22.2. Let E be a Banach space. Then the general linear group of E, $GL(E)$, is a topological group with respect to the group operator of operator composition and the topology induced by the operator norm on $\mathcal{L}(E)$.

Note. Royden and Fitzpatrick mention some special cases of the general linear group. If $E = \mathbb{R}^n$, then we know that $GL(\mathbb{R}^n)$ consists of invertible $n \times n$ matrices (we need a basis for $\mathbb{R}^n$; we can take the standard basis). The group is denoted $GL(n, \mathbb{R})$. The *special linear group* on $\mathbb{R}^n$, denoted $SL(n, \mathbb{R})$, consists of all $n \times n$ invertible matrices with determinant 1 (recall that $\det(AB) = \det(A)\det(B)$).

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