Chapter 3. Lebesgue Measurable Functions

Note. In the chapter, we define the idea of a "measurable function." This will give us the setting in which we develop Lebesgue integration in the next chapter.

Section 3.1. Sums, Products, and Compositions

Note. We want to follow the idea of Riemann sums and introduce the idea of a Lebesgue sum of "rectangles" whose heights are determined by a function and whose base is determined by the measure of a set. So in defining Lebesgue integrals, we can only consider functions which will give rectangles with measurable bases. With this as motivation, we consider the following.

Proposition 3.1. Let the function f have a measurable domain E. The following are equivalent:

(i) For each $c \in \mathbb{R}$, $\{x \in E \mid f(x) > c\} \in \mathcal{M}$.

(ii) For each $c \in \mathbb{R}$, $\{x \in E \mid f(x) \geq c\} \in \mathcal{M}$.

(iii) For each $c \in \mathbb{R}$, $\{x \in E \mid f(x) < c\} \in \mathcal{M}$.

(iv) For each $c \in \mathbb{R}$, $\{x \in E \mid f(x) \leq c\} \in \mathcal{M}$.

Each of these properties implies that for each extended real number c, $\{x \in E \mid f(x) = c\} \in \mathcal{M}$.

Definition. An extended real-valued function f defined on $E \in \mathcal{M}$ is (Lebesgue) *measurable* if it satisfies (i)–(iv) of Proposition 3.1.

Proposition 3.2. Let f be defined on $E \in \mathcal{M}$. Then f is measurable if and only if for each open $\mathcal{O}$, the inverse image of $\mathcal{O}$, $f^{-1}(\mathcal{O})$, is measurable.

Note. Proposition 3.2 makes the proof of the following relatively easy. Notice that Proposition 3.3 implies that continuous functions are measurable. Ultimately, we will show that Lebesgue integration is a generalization of Riemann integration. We know that continuous functions are Riemann integrable. We will define the Lebesgue integral of all measurable functions (when we avoid an $\infty - \infty$ meaningless computation), so knowing that continuous functions are measurable will help us establish that Riemann integrability implies Lebesgue integrability.

Proposition 3.3. A real-valued function that is continuous on its measurable domain is measurable.

Note. Recall that a monotone function can have at most countably many discontinuities (see my online notes on Analysis 1 [MATH 4217/5217] on 4.2. Monotone and Inverse Functions; see Theorem 4.14). This is useful in the proof of the following result, which is left as Problem 3.24.

Proposition 3.4. A monotone function that is defined on an interval is measurable.

Proposition 3.5. Let f be an extended real-valued function defined on E.

(i) If f is measurable on E and $f = g$ a.e., then g is measurable on E.

(ii) For $D \subseteq E$, $D \in \mathcal{M}$, f is measurable on E if and only if f restricted to D is measurable and f restricted to $E \setminus D$ is measurable.

Proposition 3.6. Suppose f, g are measurable on E and f, g are extended real-valued functions which are finite a.e. on E. Then

(i) $\alpha f + \beta g$ is measurable on E for all $\alpha, \beta \in \mathbb{R}$.

(ii) fg is measurable on E.

Note. There are two real-valued measurable functions defined on $\mathbb{R}$ whose composition is not measurable. See the example on page 57 of the text book which uses the Cantor-Lebesgue Function. In the next result, we consider a composition but we require continuity and a domain of all of $\mathbb{R}$ for one of the functions in the composition.

Proposition 3.7. Let g be a measurable real-valued function defined on E and f a continuous real-valued function defined on all of $\mathbb{R}$. The composition $f \circ g$ is measurable on E.

Note. Proposition 3.7 implies that for measurable f, $|f|$ and $|f|^p$ ($p > 0$) are measurable. We will consider integrals of $|f|$ in Section 4.4 (The General Lebesgue Integral). We will consider integrals of $|f|^p$ (usually where $p \geq 1$) extensively in Chapters 7 and 8 when we consider L^p spaces.

Proposition 3.8. For a finite family $\{f_k\}_{k=1}^n$ of measurable functions with common domain E, the functions $\max\{f_1, f_2, \ldots, f_n\}$ and $\min\{f_1, f_2, \ldots, f_n\}$ (defined pointwise) are measurable.

Note. Proposition 3.8 can be extended to the sup and inf of a countable collection of measurable functions (see Problem 3.21). It can also be used to show that each of the following are measurable:

$$\begin{aligned} |f| &= \max\{f(x), -f(x)\} \\ f^+ &= \max\{f(x), 0\} \\ f^- &= \max\{-f(x), 0\} \end{aligned}$$

(each defined pointwise). In Section 4.4 (The General Lebesgue Integral) we will define integrals of functions f in terms of integrals of f^+ and f^-.

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