Section 3.2. Sequential Pointwise Limits and Simple Approximation

Note. The following ideas concerning the convergence of a sequence of functions should be familiar from senior-level analysis and maybe even calculus.

Definition. For a sequence $\{f_n\}$ of functions with common domain E, a function f on E and a subset A of E, we say that:

(i) The sequence $\{f_n\}$ converges to f *pointwise* on A provided $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in A$.

(ii) The sequence $\{f_n\}$ converges to f *pointwise a.e.* on A provided it converges to f pointwise on $A \setminus B$, where $m(B) = 0$.

(iii) The sequence $\{f_n\}$ converges to f *uniformly* on A provided for each $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $|f - f_n| < \varepsilon$ on A for all $n \geq N$.

Note. We now address the pointwise limit of a sequence of measurable functions.

Proposition 3.9. Let $\{f_n\}$ be a sequence of measurable functions on E that converges pointwise a.e. on E to function f. Then f is measurable.

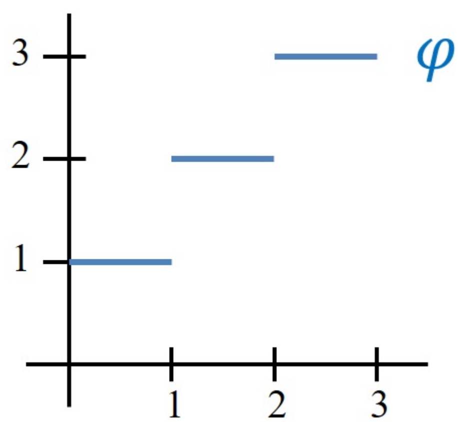
Definition. The *characteristic function* on a set A is

$$\chi_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A. \end{cases}$$

Definition. A real-valued function φ defined on a measurable set E is *simple* if it is measurable and takes on only a finite number of values. If the distinct values the function assumes are $\{c_1, c_2, \ldots, c_n\}$ and $E_k = \{x \in E \mid \varphi(x) = c_k\}$, then the *canonical representation* of φ is

$$\varphi(x) = \sum_{k=1}^{n} c_k \chi_{E_k}(x).$$

Example. Consider the simple function:



The canonical representation of φ is the obvious: $\varphi(x) = 1\chi_{[0,1)} + 2\chi_{[1,2)} + 3\chi_{[2,3)}$. Another representation is $\varphi(x) = 1\chi_{[0,3)} + 1\chi_{[1,2)} + 2\chi_{[2,3)}$, but this representation is not the canonical one since the coefficients of the characteristics functions are not distinct and the sets in the characteristic functions are not disjoint.

Note. In Chapter 4, we will develop Lebesgue integration by defining it for increasingly more general classes of functions. We will start with the class of simple functions (which will give us something like "Lebesgue sums"). When dealing with some other types of functions, we will define the Lebesgue integral by approximating functions with simple functions. Thus we consider the next two results.

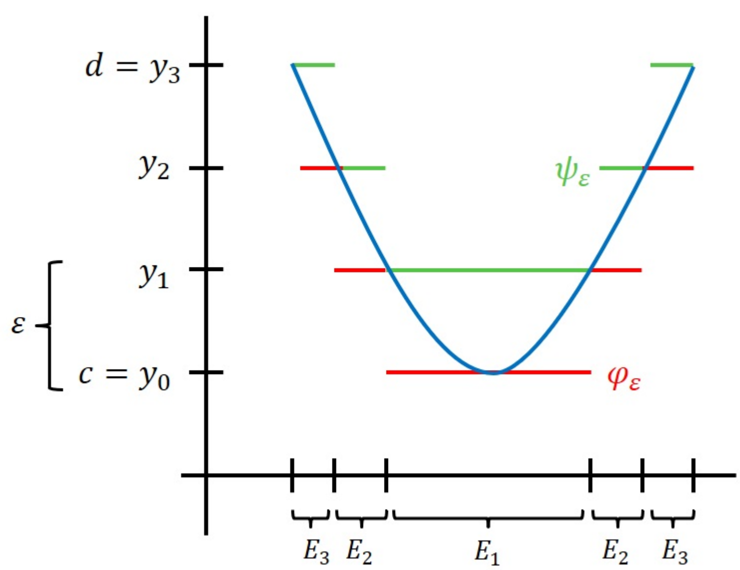
The Simple Approximation Lemma.

Let f be measurable and real valued on set E. Assume f is bounded on E (i.e., $|f| \leq M$ on E for some M). Then for each $\varepsilon > 0$, there are simple functions φ_ε and ψ_ε for which

$$\varphi_\varepsilon \leq f \leq \psi_\varepsilon \text{ and } 0 \leq \psi_\varepsilon - \varphi_\varepsilon < \varepsilon \text{ on } E.$$

(That is, these inequalities hold pointwise for each $x \in E$.)

Note. The idea behind the proof of The Simple Approximation Lemma is as given in the following figure:



Note. The next result gives a condition equivalent to the measurability of a function in terms of approximations of its relationship to a sequence of simple functions.

The Simple Approximation Theorem. An extended real-valued function f on a measurable set E is measurable if and only if there is a sequence $\{\varphi_n\}$ of simple functions on E which converges pointwise on E to f and satisfies $|\varphi_n| \leq |f|$ on E for all n. If f is nonnegative, we may choose $\{\varphi_n\}$ to be increasing.

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