Section 3.3. Littlewood's Three Principles, Egoroff's Theorem, and Lusin's Theorem

Note. Notice the quote in Royden and Fitzpatrick on page 64 to the effect:

1. Every measurable set is "nearly" a finite union of intervals.

2. Every measurable function is "nearly" continuous.

3. Every convergent sequence of measurable functions is "nearly" uniformly convergent.

These are known as *Littlewood's Three Principles* after John Edensor Littlewood (1885–1977). Quoting from the Saint Andrews Math History Webpage: "Almost all of Littlewood's mathematical research was in classical analysis, but in this area he looked at a remarkable range of subjects and he used an even broader range of techniques in proving his results. For 35 years he collaborated with G. H. Hardy working on the theory of series, the Riemann zeta function, inequalities, and the theory of functions" (from MacTutor History of Mathematics Archive biography of Littlewood, accessed 10/18/2020). Littlewood is in my mathematical genealogy six generations back. At one point in his graduate school career, his Ph.D. advisor, E. W. Barnes, gave him the problem of proving the Riemann Hypothesis!

John Edensor Littlewood (1885–1977)

From the MacTutor History of Mathematics Archive webpage.

Note. More formally, Littlewood's Three Principles are:

1. Let E be a measurable set of finite outer measure. Then for each $\varepsilon > 0$, there is a finite disjoint collection of open intervals $\{I_k\}_{k=1}^n$ for which if $\mathcal{O} = \cup_{k=1}^n I_k$, then
$$m(E \setminus \mathcal{O}) + m(\mathcal{O} \setminus E) = m(E \triangle \mathcal{O}) < \varepsilon$$
where "$\triangle$" represents the symmetric difference between two sets: $A \triangle B = (A \setminus B) \cup (B \setminus A)$. This is Theorem 2.12 with outer measure replaced by Lebesgue measure.

2. Let f be a real-valued measurable function on E. Then for each $\varepsilon > 0$, there is a continuous function g on $\mathbb{R}$ and a closed set F contained in E for which
$$f = g \text{ on } F \text{ and } m(E \setminus F) < \varepsilon.$$
This is Lusin's Theorem from page 66.

3. Assume E has finite measure. Let $\{f_n\}$ be a sequence of measurable functions on E that converges pointwise on E to the real-valued function f. Then for each $\varepsilon > 0$, there is a closed set F contained in E for which

$$\{f_n\} \to f \text{ uniformly on } F \text{ and } m(E \setminus F) < \varepsilon.$$

This is Egoroff's Theorem from page 64.

Note. Each of Littlewood's Principles involves (roughly) behavior "except on a set of measure ε." With this idea informally defined as the term "nearly," we can paraphrase Littlewood's Three Principles as:

1. Let E be a measurable set of finite outer measure. Then E is "nearly" a finite disjoint union of open intervals.

2. Let f be a real-valued measurable function on E. Then f is "nearly" continuous.

3. Let $\{f_n\}$ be a sequence of measurable functions on set E of finite measure that converges pointwise on E to real-valued function f. Then $\{f_n\}$ "nearly" converges uniformly to f.

This is still a bit informal (Principle 1 does not involve ignoring a set of measure less than ε, instead involves a symmetric difference less than ε, for example). Littlewood himself gives an informal description, as spelled out in our text on page 64 in the opening paragraph for this section (as mentioned above). A similar idea was encountered in Section 3.2 with Problems 3.16, 3.17, and 3.18. To paraphrase these problems, we have (respectively) that a characteristic function on a bounded measurable set is "nearly" a step function, a simple function on a bounded measurable set is "nearly" a step function, and a bounded measurable function on a bounded measurable set is "nearly" a step function. We now turn our attention to the proofs of the second and third principles of Littlewood.

Lemma 3.10. Assume E has finite measure. Let $\{f_n\}$ be a sequence of measurable functions on E that converges pointwise on E to the real-valued function f. Then for each $\eta > 0$ and $\delta > 0$, there is a measurable subset A of E and an index N for which

$$|f_n - f| < \eta \text{ on } A \text{ for all } n \geq N \text{ and } m(E \setminus A) < \delta.$$

Egoroff's Theorem. Assume E has finite measure. Let $\{f_n\}$ be a sequence of measurable functions on E that converges pointwise on E to the real-valued function f. Then for each $\varepsilon > 0$, there is a closed set F contained in E for which

$$\{f_n\} \to f \text{ uniformly on } F \text{ and } m(E \setminus F) < \varepsilon.$$

Note. Egoroff's Theorem does not hold (in general) if E is of infinite measure, as shown in Problem 3.27. However, the hypotheses can be weakened to pointwise a.e. convergence to f where f is finite a.e. (Problem 3.28).

Note. The following is a special case of Littlewood's Second Principle and only applies to simple functions. This result will be used in the proof of the general Second Principle (also known as "Lusin's Theorem").

Proposition 3.11. Let f be a simple function defined on E. Then for each $\varepsilon > 0$, there is a continuous function g on $\mathbb{R}$ and a closed set F contained in E for which

$$f = g \text{ on } F \text{ and } m(E \setminus F) < \varepsilon.$$

Lusin's Theorem. Let f be a real-valued measurable function on E. Then for each $\varepsilon > 0$, there is a continuous function g on $\mathbb{R}$ and a closed set F contained in E for which

$$f = g \text{ on } F \text{ and } m(E \setminus F) < \varepsilon.$$

Note. Lusin's Theorem still holds if the hypotheses are weakened from "real-valued f" to "f is finite a.e." (Problem 3.30).

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