Chapter 4. Lebesgue Integration

Section 4.1. Riemann Integral

Note. In this brief section, we define the Riemann integral of f on $[a, b]$ (when it exists) in two equivalent ways. The first way is probably what you saw in senior-level Analysis 2 (MATH 4227/5227) and the second way will act as inspiration of our approach to Lebesgue integration.

Definition. A *partition* of the interval $[a, b]$ is a set $P = \{x_0, x_1, \ldots, x_n\} \subset [a, b]$ such that $a = x_0 < x_1 < \cdots < x_n = b$.

Definition. Let f be a bounded function on $[a, b]$. For P a partition of $[a, b]$, the *lower and upper Darboux sums* for f with respect to P are

$$L(f, P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \text{ and } U(f, P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}),$$

respectively, where for $1 \le i \le n$

$$m_i = \inf\{f(x) \mid x_{i-1} < x < x_i\} \text{ and } M_i = \sup\{f(x) \mid x_{i-1} < x < x_i\}.$$

Definition. Let f be a bounded function on $[a,b]$. The *lower and upper Riemann integrals* of f over $[a,b]$ are

$$(R)\underline{\int_a^b} f = \sup\{L(f,P) \mid P \text{ is a partition of } [a,b]\},$$

and

$$(R)\overline{\int_a^b} f = \inf\{U(f,P) \mid P \text{ is a partition of } [a,b]\}.$$

Definition. Let f be a bounded function on $[a,b]$. The f is *Riemann integrable* on $[a,b]$ if $(R)\underline{\int_a^b} f = (R)\overline{\int_a^b} f$.

Note. These are the same definitions as used by J.R. Kirkwood in *An Introduction to Analysis*, 2nd Edition, Waveland Press (2002), with the exception that m_i and M_i are defined using $[x_{i-1}, x_i]$ and the term "Riemann sum" replaces the term "Darboux sum." See my online notes for Analysis 2 (MATH 4227/5227) on 6.1. The Riemann Integral based on Kirkwood's book, and the notes for this class on The Riemann-Lebesgue Theorem.

Definition. A real-valued function ψ defined on $[a,b]$ is a *step function* if there is a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a,b]$ and numbers $c_1, c_2, \ldots, c_n$ such that for $1 \le i \le n$, $\psi(x) = c_i$ if $x_{i-1} < x < x_i$.

Note. Notice that $L(\psi, P) = \sum_{i=1}^{n} c_i(x_i - x_{i-1}) = U(\psi, P)$ for all partitions P of $[a, b]$. So ψ is Riemann integrable on $[a, b]$ and $(R)\int_a^b \psi = \sum_{i=1}^{n} c_i(x_i - x_{i-1})$. So we can also express lower and upper Riemann integrals as

$$(R)\underline{\int_a^b} f = \sup\left\{(R)\int_a^b \varphi \mid \varphi \text{ is a step function and } \varphi \leq f \text{ on } [a,b]\right\}$$

and

$$(R)\overline{\int_a^b} f = \inf\left\{(R)\int_a^b \varphi \mid \varphi \text{ is a step function and } f \leq \varphi \text{ on } [a,b]\right\},$$

respectively. This idea of approximating f from below with "easy to integrate functions" and taking a supremum, and approximating f from above with "easy to integrate functions" and taking an infimum, will be the approach we take with Lebesgue integration in the next section.

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