Section 4.2. Lebesgue Integration of a Bounded Measurable Function over a Set of Finite Measure

Note. In Sections 4.2 through 4.4, we develop Lebesgue integration of measurable functions. We do so by defining the Lebesgue integral for a class of functions and then use that class of functions to define the Lebesgue integral for the next (larger) class of functions. In this section, we start with the class of *simple functions* (think of this as "Class 0"). We then use integrals of simple functions to define the Lebesgue integral of *bounded measurable functions over sets of finite measure* (think of this as "Class 1"). In Class 1, we have that everything with which we compute is bounded (namely, function values are bounded and measures of sets are bounded). In Section 4.3 we consider *nonnegative functions* (functions in "Class 2"). We define the Lebesgue integral of Class 2 functions in terms of integrals of Class 1 functions. In Class 2, we may encounter unbounded integrals (i.e., integrals which are infinite), but we will not encounter any sort of $\infty - \infty$ situation since nothing is negative here. In Section 4.4 we consider *integrable functions* (that is, functions that have a positive part and a negative part, both of which are nonnegative by definition, and both of which have a finite Lebesgue integral); these are the functions in "Class 3." The positive and negative parts of these functions are in Class 2 and have finite integrals. We then define the integral of a Class 3 function as the integral of the positive part minus the integral of the negative part (since both of these integrals in finite, we have avoided an $\infty - \infty$ situation).

Note. In Section 4.1 we saw that we could define the Riemann integral of a bounded function on a closed and bounded interval in terms of lower and upper Riemann integrals:

$$(R)\underline{\int_a^b} f = \sup\left\{(R)\int_a^b \varphi \mid \varphi \text{ is a step function and } \varphi \leq f \text{ on } [a,b]\right\}$$

and

$$(R)\overline{\int_a^b} f = \inf\left\{(R)\int_a^b \varphi \mid \varphi \text{ is a step function and } \varphi \geq f \text{ on } [a,b]\right\},$$

respectively. When the upper and lower Riemann integrals are equal, the Riemann integral is the common value of these two integrals. We now follow a similar procedure in the development of the Lebesgue integral by replacing the step functions of Riemann with simple function here. Therefore, simple functions play the same role in Lebesgue integration as step functions play in Riemann integration.

Definition. For simple function ψ on a set of finite measure E with canonical representation $\psi = \sum_{i=1}^n a_i \chi_{E_i}$, define the (Lebesgue) integral

$$\int_E \psi = \sum_{i=1}^n a_i m(E_i).$$

Note. We have defined the Lebesgue integral in terms of the canonical representation. In the next result we give the value of the Lebesgue integral of simple φ in the event that the representation of φ may not be the canonical one.

Lemma 4.1. Let $\{E_i\}_{i=1}^n$ be a finite disjoint collection of measurable subsets of a set of finite measure E. For $1 \le i \le n$, let a_i be a real number. If $\varphi = \sum_{i=1}^n a_i \chi_{E_i}$ on E then $\int_E \varphi = \sum_{i=1}^n a_i m(E_i)$.

Note. The following is the first time (of four times; once for each "Class" of functions) we'll see a linearity and monotonicity result.

Proposition 4.2. Linearity and Monotonicity of Integration.
Let φ and ψ be simple functions defined on a set of finite measure E. Then for any α, β

$$\int_E (\alpha\varphi + \beta\psi) = \alpha \int_E \varphi + \beta \int_E \psi$$

and if $\varphi \le \psi$ on E then $\int_E \varphi \le \int_E \psi$.

Note. We are ready to define the lower and upper Lebesgue integrals for Class 1 functions. Notice that the definition is very similar to the definition of the lower and upper Riemann integral of a bounded function on a closed and bounded interval, but that simple functions are used here whereas step functions were used for Riemann integrals.

Definition. Let f be a bounded function on a set of finite measure. Define the *lower Lebesgue integral*

$$\underline{\int_E} f = \sup\left\{ \int_E \varphi \,\middle|\, \varphi \text{ is simple, } \varphi \le f \right\}.$$

Define the *upper Lebesgue integral*

$$\overline{\int_E} f = \inf \left\{ \int_E \psi \,\middle|\, \psi \text{ is simple, } \psi \geq f \right\}.$$

Note. By monotonicity, since f is bounded, both $\underline{\int_E} f$ and $\overline{\int_E} f$ are finite. Also by monotonicity $\underline{\int_E} f \leq \overline{\int_E} f$.

Definition. A bounded function f on a domain E of finite measure is said to be *Lebesgue integrable* over E provided $\underline{\int_E} f = \overline{\int_E} f$. The common value is the *Lebesgue integral* of f over E, denoted $\int_E f$.

Note. We'll see below that all Class 1 functions are Lebesgue integrable (see Theorem 4.4).

Note. Recall from the Riemann-Lebesgue Theorem (Theorem 6-11 in the Riemann-Lebesgue Theorem supplement) that a bounded function on $[a, b]$ is Riemann integrable if and only if f is continuous almost everywhere. By Proposition 3.3 a real-valued function that is continuous on its measurable domain is a measurable function, and by Proposition 3.5(i) if f is measurable on E and $f = g$ a.e. on E then g is measurable on E. So a bounded function that is continuous a.e. on $[a, b]$ is measurable and so both the Riemann integral and the Lebesgue integral are defined for such a function. We now show that these integrals have the same values, as expected.

Theorem 4.3. Let f be a bounded function defined on $[a, b]$. If f is Riemann integrable over $[a, b]$ then it is Lebesgue integrable over $[a, b]$ and the two integrals are equal.

Note. The following result establishes the Lebesgue integrability of the class of functions studied in this section (the "Class 1" functions).

Theorem 4.4. Let f be a bounded measurable function on a set of finite measure E. Then f is integrable on E.

Note. The converse of Theorem 4.4 is also true. That is, if bounded f on set of finite measure E is integrable (i.e., $\int_E f$ exists), then f is measurable (see Theorem 5.7 in 5.3. Characterization of Riemann and Lebesgue Integrability).

Note. We now have our second encounter with a linearity and monotonicity result. Proposition 4.2 addressed linearity and monotonicity for simple functions (i.e., Class 0 functions) and the next result addresses bounded functions on sets of finite measure (i.e., Class 1 functions).

Theorem 4.5. Linearity and Monotonicity.

Let f and g be bounded measurable functions on a set of finite measure E. Then for all $\alpha, \beta \in \mathbb{R}$

$$\int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g.$$

Moreover, if $f \leq g$ on E, then $\int_E f \leq \int_E g$.

Note. Linearity and and monotonicity on Class 1 can be used to prove the following two corollaries.

Corollary 4.6. Let f be a bounded measurable function on a set E of finite measure. Suppose A and B are measurable disjoint subsets of E. Then

$$\int_{A \dot{\cup} B} f = \int_A f + \int_B f.$$

Corollary 4.7. Let f be a bounded measurable function on a set of finite measure E. Then

$$\left| \int_E f \right| \leq \int_E |f|.$$

Note. Recall that a sequence $\{f_n\}$ of Riemann integrable functions on $[a, b]$ which converges uniformly to f on $[a, b]$ satisfies

$$\lim_{n\to\infty} \left((R) \int_a^b f_n(x)\, dx \right) = (R) \int_a^b \left(\lim_{n\to\infty} f_n(x) \right) dx = (R) \int_a^b f(x)\, dx$$

(see Theorem 8-3 of the Riemann-Lebesgue Theorem handout). The next proposition shows that a similar result holds for Class 1 functions and Lebesgue integration.

Proposition 4.8. Let $\{f_n\}$ be a sequence of bounded measurable functions on a set of finite measure on E. If $\{f_n\} \to f$ uniformly on E, then

$$\lim_{n\to\infty}\left(\int_E f_n\right) = \int_E \left(\lim_{n\to\infty} f_n\right) = \int_E f.$$

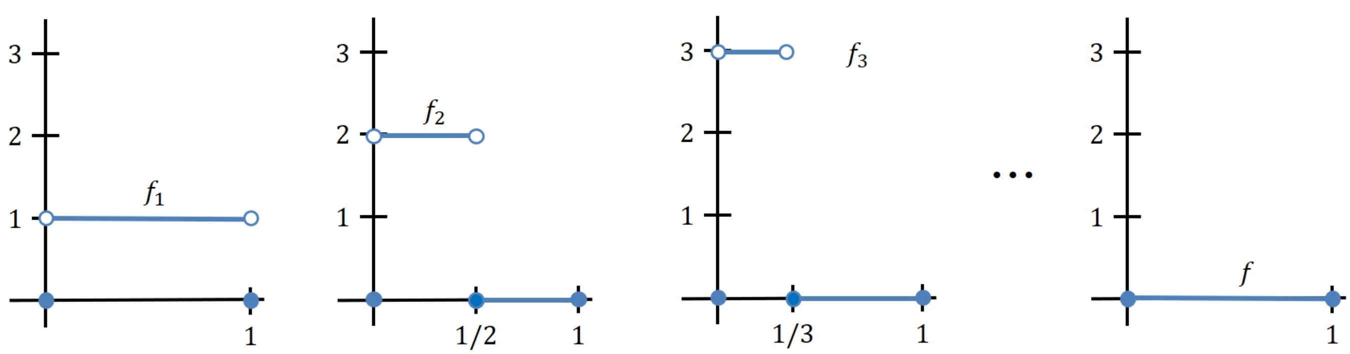
Note. We are interested in general in when $\int_E f = \int_E(\lim f_n) = \lim(\int_E f_n)$ for a sequence $\{f_n\}$ of measurable functions on set E which converges to measurable function f pointwise on E. When this result holds, we have a "convergence theorem." We will have three such results, one for each of Class 1, Class 2, and Class 3 of functions. The next note shows that for Class 1 functions we need some additional hypothesis on the sequence $\{f_n\}$.

Note 4.2.A. Define f_n on $[0,1]$ as:

$$f_n(x) = \begin{cases} 0 \text{ if } x = 0 \text{ or } x \in [1/n, 1] \\ n \text{ if } x \in (0, 1/n). \end{cases}$$

Then $f_n \to f \equiv 0$ (pointwise but not uniformly) and $\displaystyle\int_{[0,1]} f_n = 1$ for all $n \in \mathbb{N}$. However,

$$\lim_{n\to\infty}\left(\int_{[0,1]} f_n\right) = 1 \neq \int_{[0,1]} \left(\lim_{n\to\infty} f_n\right) = \int_{[0,1]} 0 = 0.$$



Note. To prove our first convergence theorem, we need Egoroff's Theorem:

> **Egoroff's Theorem.** Assume E has finite measure. Let $\{f_n\}$ be a sequence of measurable functions on E that converges pointwise on E to real-valued function f. Then for every $\varepsilon > 0$, there is a closed set $F \subset E$ for which $\{f_n\} \to f$ uniformly on F and $m(E \setminus F) < \varepsilon$.

See 3.3. Littlewood's Three Principles, Egoroff's Theorem, and Lusin's Theorem.

Note. To deal with the example given in Note 4.2.A, we impose a condition of "uniform pointwise boundedness" which insures that each function in the sequence $\{f_n\}$ is bounded by the same constant. Notice that this is *not* the case for the example given in Note 4.2.A.

Bounded Convergence Theorem.

Let $\{f_n\}$ be a sequence of measurable functions on a set of finite measure E. Suppose $\{f_n\}$ is uniformly pointwise bounded on E, that is, there is a number $M \geq 0$ for which $|f_n| \leq M$ on E for all n. If $\{f_n\} \to f$ pointwise on E, then

$$\lim_{n\to\infty}\left(\int_E f_n\right) = \int_E \left(\lim_{n\to\infty} f_n\right) = \int_E f.$$

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