Section 4.3. The Lebesgue Integral of a Measurable Nonnegative Function

Note. In this section we drop the restrictions of boundedness, but only consider *nonnegative* (possibly extended real number valued) functions; that is, we consider my so-called "Class 2" functions. Since we have lost boundedness, we may have integrals which are infinite. Notice in this class we have no concerns over $\infty - \infty$ computations since everything with which we compute (namely function values and measures of sets) is nonnegative.

Definition. A measurable function f on set E is of *finite support* if there is $E_0 \subset E$ for which $m(E_0) < \infty$ and $f \equiv 0$ on $E \setminus E_0$. The *support* of f is the set $\{x \in E \mid f(x) \neq 0\}$.

Note. In Class 1, which we considered in the previous section, everything with which we compute (namely function values and measures of sets) is bounded. We first drop the condition on the finiteness of the measure of a set over which we integrate, but we impose a condition of boundedness and finite support. This allows us to consider certain Class 2 functions which are *effectively* in Class 1. To be formal, we have to define the integral of such functions separately and we do so now.

Definition. Let f be a bounded measurable function on E which has finite support, say $f \equiv 0$ on $E \setminus E_0$ and $m(E_0) < \infty$. Define $\int_E f = \int_{E_0} f$.

Note. In Problem 4.18 you are asked to verify that the previous definition is "proper" (i.e., the integral is well-defined). That is, the definition is independent of the choice of set E_0 of finite measure.

Note. We now define the integral of Class 2 functions (i.e., nonnegative functions) in terms of Class 1 functions (or, at least, in terms of functions which are *effectively* in Class 1).

Definition. For f a nonnegative measurable function on E, we define the integral of f over E as

$$\int_E f = \sup\{\textstyle\int_E h \mid h \text{ is bounded, measurable, of finite support, and } 0 \leq h \leq f\}.$$

Note. It might appear that, when integrating a function that is ∞ on a set of measure 0, we are using some sort of idea that $0 \cdot \infty = 0$. WE ARE NOT!!! We have hidden any such problem in the stages of development leading to the previous definition. Namely, in Problem 4.17 (which uses the previous definition and Problem 4.9) it is to be shown that if $f \equiv \infty$ on E and $m(E) = 0$ then $\int_E f = 0$. Surprisingly, the 3rd edition of Royden's *Real Analysis* states: "The operation $\infty - \infty$ is left undefined, but we shall adopt the arbitrary *convention* that $0 \cdot \infty = 0$." (See page 36 of the 3rd edition.) In the 4th edition, Royden and Fitzpatrick deal with the arithmetic of extended real numbers on page 10—notice there is no mention of $0 \cdot \infty$ nor $\infty - \infty$ (since both are now left, appropriately, undefined).

Note. Another setting in which we might encounter a $0 \cdot \infty$ problem involves a function $f \equiv 0$ on a set E of infinite measure. The previous definition gives us:

Lemma 4.3.A. If $f \equiv 0$ on measurable set E where $m(E) = \infty$, then $\int_E f = 0$.

This holds since the only bounded measurable functions h of finite support with $0 \le h \le f = 0$ on E are the constant functions equal to 0 on measurable subsets of E of finite measure. Since constant $h \equiv 0$ is a bounded nonnegative function of finite support, we then have by the definition above that for each such h, $\int_E h = 0$. Hence $\int_E f = 0$.

Note. Now if nonnegative function f defined on set E satisfies $f = 0$ a.e. on E, then certainly $\int_E f = 0$. We use the next result to show that the converse of this observation also holds.

Chebychev's Inequality.

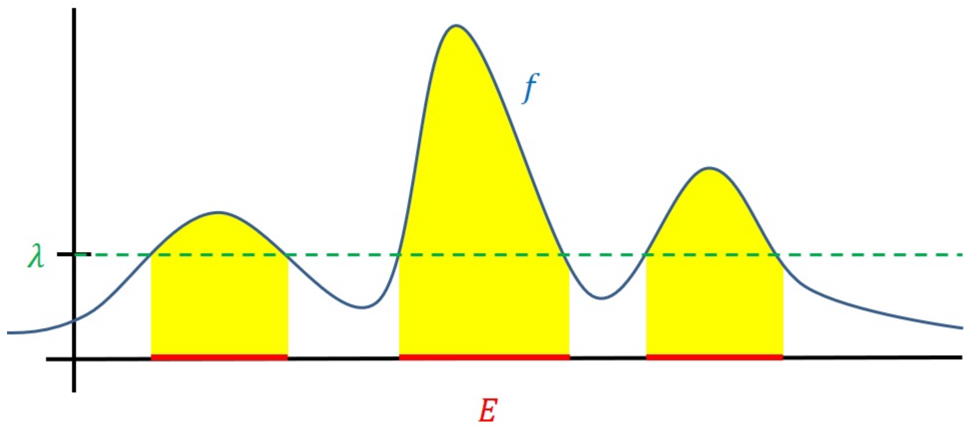
Let f be a nonnegative measurable function on E. Then for any $\lambda > 0$,

$$m\left(\{x \in E \mid f(x) \ge \lambda\}\right) \le \frac{1}{\lambda} \int_E f.$$

Note. Chebychev's Inequality is not surprising when we write

$$\lambda \cdot m\left(\{x \in E \mid f(x) \ge \lambda\}\right) \le \int_E f$$

and consider the following illustration:



It allows us to prove the following.

Proposition 4.9. Let f be a nonnegative measurable function on set E. Then $\int_E f = 0$ if and only if $f = 0$ a.e. on E.

Note. Now for linearity and monotonicity for Class 2 functions. This is our third (of four) such results.

Theorem 4.10. Linearity and Monotonicity of Integration.
Let f and g be nonnegative measurable functions on E. Then for any $\alpha > 0$ and $\beta > 0$,

$$\int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g.$$

Moreover, if $f \leq g$ on E then $\int_E f \leq \int_E g$.

Note. Just as we had an additivity result for Class 1 functions (Corollary 4.6), we have a similar result for Class 2 functions.

Theorem 4.11. Additivity Over Domain of Integration.

Let f be a nonnegative measurable function on E. If A and B are disjoint measurable subsets of E, then

$$\int_{A \cup B} f = \int_A f + \int_B f.$$

In particular, if E_0 is a subset of E of measure zero, then $\int_E f = \int_{E \setminus E_0} f$.

Note. We are interested in a convergence theorem for Class 2 functions (i.e., nonnegative functions), similar to the Bounded Convergence Theorem for Class 1 functions. The next result is a step in this direction.

Fatou's Lemma. Let $\{f_n\}$ be a sequence of nonnegative measurable functions on E. If $\{f_n\} \to f$ pointwise a.e. on E, then

$$\int_E f = \int_E \lim f_n \le \liminf \left(\int_E f_n \right).$$

Note 4.3.A. See page 83 for two examples that show that the inequality in Fatou's Lemma can be strict. Here is another:

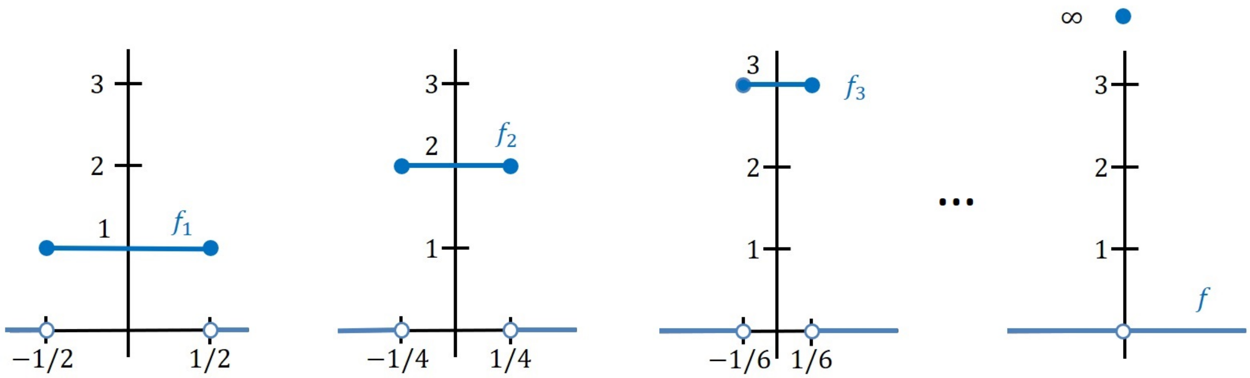
Define $f_n(x) = n \cdot \chi_{[-1/(2n),1/(2n)]}$. Then $\displaystyle\int_{\mathbb{R}} f_n = n \cdot \frac{1}{n} = 1$ and

$$\{f_n\} \to \begin{cases} 0 \text{ if } x \ne 0 \\ \infty \text{ if } x = 0, \end{cases}$$

so $\lim f_n = f = 0$ a.e. and $\int_{\mathbb{R}} f = 0$. That is

$$\int_{\mathbb{R}} f = 0 < 1 = \lim_{n \to \infty} \left(\int_{\mathbb{R}} f_n \right) = \liminf \left(\int_{\mathbb{R}} f_n \right).$$

The graphs of some of these functions are:



The fact that the integral of each f_n is 1 suggests that this example could have relevance for probability distributions. For a further discussion of this, see my supplement on The Dirac Delta Function, A Cautionary Tale.

Note. Fatou's Lemma is sort of "half" of a convergence theorem in the sense that it relates $\int_E \lim f_n$ to $\liminf \int_E f_n$, where $f_n \to f$ a.e. on E. We can still draw a certain conclusion for *any* sequence of nonnegative measurable functions on E, not just an a.e. convergent sequence. This is addressed in the Generalized Fatou's Lemma, a proof of which is required in Problem 4.27:

Generalized Fatou's Lemma. If $\{f_n\}$ is a sequence of nonnegative measurable functions on E, then

$$\int_E \liminf f_n \leq \liminf \int_E f_n.$$

Note. We see from the example given in Note 4.3.A that we need some extra conditions beyond pointwise converge to get a convergence theorem for Class 2 functions. With monotonicity in the sequence of functions and Fatou's Lemma, we can prove the following.

Monotone Convergence Theorem. Let $\{f_n\}$ be an increasing sequence of nonnegative measurable functions on E. If $\{f_n\} \to f$ pointwise a.e. on E, then

$$\lim_{n\to\infty} \left(\int_E f_n \right) = \int_E \left(\lim_{n\to\infty} f_n \right) = \int_E f.$$

Note. The Monotone Convergence Theorem allows us to deal with the integral of a *series* of nonnegative measurable functions which converge in terms of a series of integrals.

Corollary 4.12. Let $\{u_n\}$ be a sequence of nonnegative measurable functions on E. If $f = \sum_{n=1}^{\infty} u_n$ pointwise a.e. on E, then

$$\int_E f = \sum_{n=1}^{\infty} \left(\int_E u_n \right).$$

Note. The next definition is used when we consider in the next section our last class of functions. The term "integrable" is being used in a somewhat different way here. In the past we have used this term in connection with the *existence* of an integral (usually in terms of the equality of upper and lower integrals) and not in connection with the finiteness of the value of the integral. This finiteness will be used in the next section to avoid an $\infty - \infty$ situation when we allow functions to be both unbounded and negative.

Definition. A nonnegative measurable function f on a measurable set E is said to be *integrable* over E provided $\int_E f < \infty$.

Note. Not surprisingly the integrability of f implies some type of finiteness of f, as given next.

Proposition 4.13. Let nonnegative f be integrable over E. Then f is finite a.e. on E.

Note. Finally, we consider a convergent sequence of functions on which we impose a uniform boundedness on the integrals themselves of the functions in this sequence. This result also deserves to be considered a "convergence theorem."

Beppo Levi's Lemma. Let $\{f_n\}$ be an increasing sequence of nonnegative measurable functions on E. If the sequence $\{\int_E f_n\}$ is bounded, then $\{f_n\}$ converges pointwise on E to a measurable function f that is finite a.e. on E and

$$\lim_{n\to\infty}\left(\int_E f_n\right) = \int_E f < \infty.$$

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