Section 4.4. The General Lebesgue Integral

Note. In this section, we define our last class of functions. We drop all conditions on measurable function f except for a type of boundedness that will allow us to avoid the meaningless "$\infty - \infty$" in our computations.

Definition. For extended real-valued function f on E, define

$$f^+(x) = \max\{f(x), 0\} \text{ and } f^-(x) = \max\{-f(x), 0\}$$

for all $x \in E$. We call f^+ the *positive part* of f and we call f^- the *negative part* of f.

Note. Notice that the positive part and negative part of f are both nonnegative. We have $f = f^+ - f^-$ and $|f| = f^+ + f^-$. Recall from the previous section that a nonnegative function is "integrable" on set E if $\int_E f < \infty$. The next result relates the integrabiity of the nonnegative functions f^+, f^-, and $|f|$.

Proposition 4.14. Let f be a measurable function on E. Then f^+ and f^- are integrable over E if and only if $|f|$ is integrable over E.

Note. We now extend the definition of "integrable" (in the setting that it was defined in the previous section, namely for nonnegative functions) to the setting of general measurable functions.

Definition. A measurable function f on E is said to be *integrable* over E provided $|f|$ is integrable over E. In this case, define the integral of f over E by

$$\int_E f = \int_E f^+ - \int_E f^-.$$

Note. By Proposition 4.14 we see that for integrable f on set E, both $\int_E f^+$ and $\int_E f^-$ are finite so that we have avoided a computation involving $\infty - \infty$ in the previous definition. Notice that for nonnegative functions (for which, of course, $|f| = f$) the definition of integrable here coincides with the definition as stated in the previous section. Our "Class 3" functions are those measurable functions which are integrable (over a given set E). We now establish some properties of Class 3 functions.

Proposition 4.15. Let f be integrable over E. Then f is finite a.e. on E and $\displaystyle\int_E f = \int_{E \setminus E_0} f$ if $E_0 \subset E$ and $m(E_0) = 0$.

Note. The next result is similar to the Direct Comparison Test from Calculus 2 concerning the convergence of improper integrals. See Theorem 2 in my online Calculus 2 (MATH 1920) notes on 8.7. Improper Integrals.

Proposition 4.16. The Integral Comparison Test.

Let f be a measurable function on E. Suppose there is a nonnegative function g that is integrable over E and $|f| \leq g$ on E. Then f is integrable over E and $\displaystyle\left|\int_E f\right| \leq \int_E |f|$.

Note. The purpose of the next definition is to deal with the case of extended real valued functions where the value of $f + g$ could yield "$\infty - \infty$." Though this may occur at some x values, we limit it to a set of measure 0 which we then ignore in the integral of $f + g$.

Definition. Let f and g be extended real valued functions defined on E and suppose they are finite a.e. on E, say f and g are finite on $A \subset E$ where $m(E \setminus A) = 0$. Define $\int_E (f+g) = \int_A (f+g)$.

Note. We now establish our last linearity and monotonicity theorem. We prove this result for Class 3 functions.

Theorem 4.17. Linearity and Monotonicity of Integration.
Let the functions f and g be integrable over E. Then for any α and β, the function $\alpha f + \beta g$ is integrable over E and

$$\int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g.$$

Also, if $f \leq g$ on E, then $\int_E f \leq \int_E g$.

Note. We now state and prove an additivity result for Class 3 functions.

Corollary 4.18. Additivity Over Domains of Integration.

Let f be integrable over E. Assume A and B are disjoint measurable subsets of E. Then

$$\int_{A\cup B} f = \int_A f + \int_B f.$$

Note. For a convergence theorem for Class 3 functions, we impose a sort-of uniform boundedness on the sequence $\{f_n\}$. This is accomplished by bounding (or "dominating") the functions f_n, not by a constant, but by an integrable function.

Theorem. The Lebesgue Dominated Convergence Theorem.

Let $\{f_n\}$ be a sequence of measurable functions on E. Suppose there is a function g that is integrable over E and dominates $\{f_n\}$ in the sense that $|f_n| \leq g$ on E for all n. If $\{f_n\} \to f$ pointwise a.e. on E, then f is integrable over E and

$$\lim_{n\to\infty}\left(\int_E f_n\right) = \int_E \left(\lim_{n\to\infty} f_n\right) = \int_E f.$$

Note. The following generalization of the Lebesgue Dominated convergence Theorem replaces integrable g with a sequence of g_n. The proof is to be given in Problem 4.32.

Theorem 4.19. The General Lebesgue Dominated Convergence Theorem.

Let $\{f_n\}$ be a sequence of measurable functions on E that converges pointwise a.e. on E to f. Suppose there is a sequence $\{g_n\}$ of nonnegative measurable functions on E that converges pointwise a.e. on E to g and dominates $\{f_n\}$ on E in the sense that $|f_n| \leq g_n$ on E for all $n \in \mathbb{N}$. If $\lim_{n\to\infty}\left(\int_E g_n\right) = \int_E g < \infty$ then

$$\lim_{n\to\infty}\left(\int_E f_n\right) = \int_E \left(\lim_{n\to\infty} f_n\right) = \int_E f.$$

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