Section 4.5. Countable Additivity and Continuity of Integration

Note. In this section, we prove two results for Lebesgue *integrals* which parallel results of Lebesgue *measure*. These properties "have no counterpart for the Riemann integral" (page 90), unlike many of the previous results of this chapter.

Theorem 4.20. The Countable Additivity of Integration.
Let f be integrable over E and $\{E_n\}_{n=1}^\infty$ a disjoint collection of measurable subsets of E whose union is E. Then

$$\int_E f = \sum_{n=1}^\infty \left(\int_{E_n} f \right).$$

Theorem 4.21. The Continuity of Integration.
Let f be integrable over E.

(i) If $\{E_n\}_{n=1}^\infty$ is an ascending countable collection of measurable subsets of E (that is, $E_i \subset E_{i+1}$ for all $i \in \mathbb{N}$), then

$$\int_{\cup_{n=1}^\infty E_n} f = \int_{\lim_{n\to\infty} E_n} f = \lim_{n\to\infty} \left(\int_{E_n} f \right).$$

(ii) If $\{E_n\}_{n=1}^\infty$ is a descending countable collection of measurable subsets of E (that is, $E_{i+1} \subset E_i$ for all $i \in \mathbb{N}$), then

$$\int_{\cap_{n=1}^\infty E_n} f = \int_{\lim_{n\to\infty} E_n} f = \lim_{n\to\infty} \left(\int_{E_n} f \right).$$

Note. The proof of Theorem 4.21 is Exercise 4.39.

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