Section 4.6. Uniform Integrability: The Vitali Convergence Theorem

Note. In this section, we introduce a new condition on a set of functions (uniform integrability) which produces another convergence theorem that is valid on sets of finite measure. The new theorem is the Vitali Convergence Theorem, which is generalized in Section 5.1.

Lemma 4.22. Let E be a set of finite measure and $\delta > 0$. Then E is the disjoint union of a finite collection of sets, each of which has measure less than δ.

Note. The following is an ε/δ-type continuity of integrals over measurable sets (not surprising since f is assumed to be integrable over the set).

Proposition 4.23. Let f be a measurable function on E. If f is integrable over E, then for each $\varepsilon > 0$, there is $\delta > 0$ for which:

$$\text{if } A \subseteq E \text{ is measurable and } m(A) < \delta \text{ then } \int_A |f| < \varepsilon. \tag{26}$$

Conversely, for $m(E) < \infty$, if for each $\varepsilon > 0$, there is a $\delta > 0$ for which (26) holds, then f is integrable over E.

Note. The following definition requires that Proposition 4.23 hold uniformly (that is, for given $\varepsilon > 0$, there exists $\delta > 0$ that works FOR ALL...) over a set ("family") of functions.

Definition. A family $\mathcal{F}$ of measurable functions on E is *uniformly integrable* over E (also called *equiintegrable*) provided for all $\varepsilon > 0$, there is a $\delta > 0$ such that for each $f \in \mathcal{F}$:

$$\text{if } A \subseteq E \text{ is measurable and } m(A) < \delta \text{ then } \int_A |f| < \varepsilon.$$

Example. If we have a certain type of boundedness on the functions in $\mathcal{F}$, then we can get uniform integrability. Suppose g is nonnegative and integrable over E. Define the family

$$\mathcal{F} = \{f \mid f \text{ is measurable on } E \text{ and } |f| \le g \text{ on } E\}.$$

Then $\mathcal{F}$ is uniformly integrable. This follows by applying Proposition 4.23 to g and the fact that $\int_A |f| \le \int_A g$ by monotonicity.

Proposition 4.24. Let $\{f_k\}_{k=1}^n$ be a finite collection of functions, each of which is integrable over E. Then $\{f_k\}_{k=1}^n$ is uniformly integrable.

Note. We now explore properties of uniformly integrable families and give a convergence theorem for a uniformly integrable (and convergent) sequence of functions.

Proposition 4.25. Assume E has finite measure. Let the sequence of functions $\{f_k\}_{k=1}^\infty$ be uniformly integrable over E. If $\{f_n\} \to f$ pointwise a.e. on E, then f is integrable over E.

Note. Problem 4.41 shows that Proposition 4.25 does not hold for infinite measure sets E.

The Vitali Convergence Theorem.

Let E be of finite measure. Suppose sequence $\{f_n\}$ is uniformly integrable over E. If $\{f_n\} \to f$ pointwise a.e. on E, then f is integrable over E and

$$\lim_{n\to\infty}\left(\int_E f_n\right) = \int_E \left(\lim_{n\to\infty} f_n\right) = \int_E f.$$

Note. In a sense, the Vitali Convergence Theorem swaps the domination of the Lebesgue Dominated Convergence Theorem for finite measure and uniform integrability.

Note. The following result shows that uniform integrability is necessary for the Vitali Convergence Theorem (at least, in the case of a nonnegative sequence of functions which converge to the zero function).

Theorem 4.26. Let E be of finite measure. Suppose $\{h_n\}$ is a sequence of nonnegative integrable functions that converges pointwise a.e. on E to $h \equiv 0$. Then

$$\lim_{n\to\infty}\left(\int_E h_n\right) = 0 \text{ if and only if } \{h_n\} \text{ is uniformly integrable over } E.$$

Note. Problem 4.42 shows that Theorem 4.26 does not hold if we drop the assumption of nonnegativity of the h_n's.

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