Chapter 5. Lebesgue Integration: Further Topics

Section 5.1. Uniform Integrability and Tightness: A General Vitali Convergence Theorem

Note. In Section 4.6, the Vitali Convergence Theorem required a set E of finite measure. We can show that this is necessary with an example which satisfies the other hypotheses of the Vitali Convergence Theorem but which violates the conclusion. For $n \in \mathbb{N}$, define $f_n = \chi_{[n,n+1]}$ and $f \equiv 0$ on $E = \mathbb{R}$. Then $\{f_n\}$ is uniformly integrable over $\mathbb{R}$ and converges pointwise to f. However,

$$\lim_{n\to\infty}\left(\int_E f_n\right) = 1 \neq 0 = \int_E \left(\lim_{n\to\infty} f_n\right) = \int_E f.$$

In this section, we replace the condition of the finiteness of $m(E)$ with a different condition on $\{f_n\}$.

Proposition 5.1. Let f be integrable over E. Then for each $\varepsilon > 0$, there is a set of finite measure E_0 for which $\int_{E\setminus E_0} |f| < \varepsilon$.

Definition. A family $\mathcal{F}$ of measurable functions is said to be *tight* over E provided for each $\varepsilon > 0$, there is a subset $E_0 \subseteq E$ of finite measure for which

$$\int_{E\setminus E_0} |f| < \varepsilon \text{ for all } f \in \mathcal{F}.$$

Note 5.1.A. If $f \in \mathcal{F}$ where $\mathcal{F}$ is uniformly integrable and tight over E, then there is finite measure $E_0 \subset E$ such that $\int_{E \setminus E_0} |f| < \varepsilon$. By Proposition 4.23, since $m(E_0) < \infty$ and (26) holds because $\mathcal{F}$ is uniformly integrable, then f is integrable on E_0. Therefore f (and $|f|$) are integrable on E.

The General Vitali Convergence Theorem.

Let $\{f_n\}$ be a sequence of functions on E that is uniformly integrable and tight over E. Suppose $\{f_n\} \to f$ pointwise a.e. on E. Then f is integrable over E and

$$\lim_{n\to\infty}\left(\int_E f_n\right) = \int_E \left(\lim_{n\to\infty} f_n\right) = \int_E f.$$

Note. The next result is similar to Theorem 4.26, but it drops the condition of finiteness of the measure of set E and instead includes the condition of tightness on the sequence of functions. A proof of Corollary 5.2 is to be given in Problem 5.1.

Corollary 5.2. Let $\{h_n\}$ be a sequence of nonnegative integrable functions on E. Suppose $\{h_n(x)\} \to 0$ for almost all $x \in E$. Then

$$\lim_{n\to\infty}\left(\int_E h_n\right) = 0 \text{ if and only if } \{h_n\} \text{ is uniformly integrable and tight over } E.$$

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