Section 5.2. Convergence in Measure

Note. In this section, we introduce a new kind of convergence of a sequence of functions on a set. This convergence generalizes pointwise convergence and many of our results stated so far hold when pointwise convergence is replaced with convergence in measure.

Definition. Let $\{f_n\}$ be a sequence of measurable functions on E and f a measurable function on E for which f and each f_n are finite a.e. on E. The sequence $\{f_n\}$ *converges in measure* on E to f provided for each $\eta > 0$,

$$\lim_{n\to\infty} m\left(\{x \in E \mid |f_n(x) - f(x)| > \eta\}\right) = 0.$$

Note. When we say "$\{f_n\}$ converges in measure on E to f" it is implied that f and each f_n are measurable and finite a.e. on E. See page 100 of Royden and Fitzpatrick.

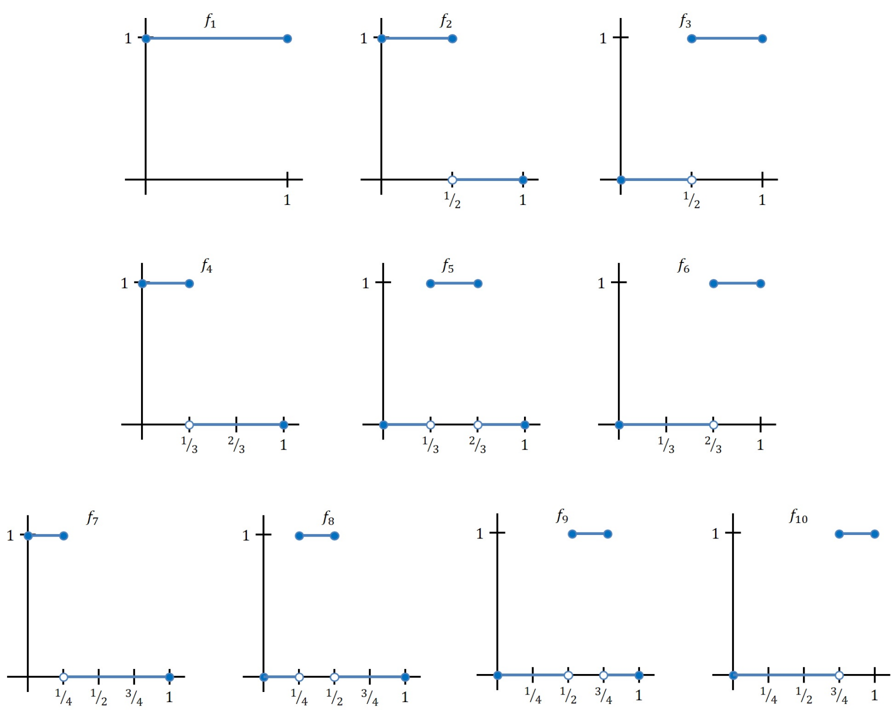
Proposition 5.3. Assume E has finite measure. Let $\{f_n\}$ be a sequence of measurable functions on E that converges pointwise a.e. on E to f, and f and each f_n are finite a.e. on E. Then $\{f_n\} \to f$ in measure on E.

Note. The following example shows that a sequence of functions may converge in measure but not converge pointwise (anywhere). So convergence in measure is a legitimate generalization of pointwise convergence. More precisely, convergence in measure is a generalization of uniform convergence on E, since uniform convergence on E implies that the set $\{x \in E \mid |f_n(x) - f(x)| > \eta\}$ is empty for n sufficiently large (and therefore uniform convergence implies convergence in measure).

Example. We cut interval $[0,1]$ into n pieces of equal length for each $n \in \mathbb{N}$. We then create a sequence of intervals $\{I_n\}_{n=1}^{\infty}$ as follows:

$$\begin{aligned}
&[0,1],\\
&[0,1/2],[1/2,1],\\
&[0,1/3],[1/3,2/3],[2/3,1],\\
&[0,1/4],[1/4,2/4],[2/4,3/4],[3/4,1],\\
&\vdots
\end{aligned}$$

Define f_n as χ_{I_n} restricted to $[0,1]$. Let $f \equiv 0$. Then $\{f_n\} \not\to f$ pointwise since for each $x \in [0,1]$, $f_n(x) = 1$ for an infinite number of $n \in \mathbb{N}$. Now for each $k \in \mathbb{N}$, if $n > 1+2+\cdots+k = k(k+1)/2$ then $m(I_n) < 1/k$, since f_n is then defined in terms of a characteristic function on an interval of length $1/k$. So for $0 < \eta < 1$, since $\{x \in E \mid |f_n(x) - f(x)| > \eta\} \subseteq I_n$, then $0 \le \lim_{n\to\infty} m(\{x \in E \mid |f_n(x) - f(x)| > \eta\}) \le \lim_{n\to\infty} m(I_n) = 0$. Therefore $\{f_n\} \to f$ in measure.



Theorem 5.4. (Riesz.)

If $\{f_n\} \to f$ in measure on E, then there is a subsequence $\{f_{n_k}\}$ that converges pointwise a.e. on E to f.

Note. The following is similar to Corollary 5.2, but involves convergence in measure instead of a.e. pointwise convergence.

Corollary 5.5. Let $\{f_n\}$ be a sequence of nonnegative integrable functions on E. Then $\lim_{n\to\infty} \int_E f_n = 0$ if and only if: $\{f_n\} \to 0$ in measure on E and $\{f_n\}$ is uniformly integrable and tight over E.

Note. Problem 5.8 really motivates convergence in measure. It states that that (1) Fatou's Lemma, (2) the Monotone Convergence Theorem, (3) the Lebesgue Dominated Convergence Theorem, and (4) the Vitali Convergence Theorem all remain true if "pointwise convergence a.e. on E" is replaced with "convergence in measure on E."

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