Section 5.3. Characterization of Riemann and Lebesgue Integrability

Note. Recall that we developed Lebesgue integration in classes:

Class 0. Simple functions (which are, by definition, measurable).

Class 1. Bounded measurable functions on sets of finite measure.

Class 2. Nonnegative measurable functions.

Class 3. Integrable measurable functions.

The definition of the integral of a Class 2 function is based on integrals of Class 1 functions, and the definition of the integral of a Class 3 function is based on integrals of Class 2 functions (provided we avoid $\infty - \infty$). Therefore, we have defined the Lebesgue Integral of any measurable function (except for the $\infty - \infty$ thing). The purpose of this section is to show that any function which is *bounded* and for which the Lebesgue integral is defined is measurable. Since Classes 2 and 3 are based on bounded measurable functions, then the result of this section is general. (Recall that the Riemann-Lebesgue Theorem—also part of this section—deals with bounded functions on closed and bounded intervals. All other types of Riemann integrals, "improper integrals," are defined in terms of these bounded integrals.)

Lemma 5.6. Let $\{\varphi_n\}$ and $\{\psi_n\}$ be sequences of measurable functions, each of which is integrable over E, such that $\{\varphi_n\}$ is increasing while $\{\psi_n\}$ is decreasing on E. Let the function f on E have the property $\varphi_n \le f \le \psi_n$ on E for all n. If $\lim_{n\to\infty}\left(\int_E(\psi_n - \varphi_n)\right) = 0$, then $\{\varphi_n\} \to f$ pointwise a.e. on E, $\{\psi_n\} \to f$ pointwise a.e. on E, f is integrable over E,

$$\lim_{n\to\infty}\left(\int_E \varphi_n\right) = \int_E f \text{ and } \lim_{n\to\infty}\left(\int_E \psi_n\right) = \int_E f.$$

Note. The following result deals with necessary and sufficient conditions for "Lebesgue integrability." That is, the existence of the Lebesgue integral, as opposed to the concept of "integrability" introduced in Section 4.3 which concerns finiteness of integrals of nonnegative functions.

Theorem 5.7. Let f be a bounded function on a set of finite measure E. Then f is Lebesgue integrable over E if and only if f is measurable.

Note. We have already encountered the following in our motivation of Lebesgue measure and integration. We called it the Riemann-Lebesgue Theorem, but Royden and Fitzpatrick label it "Lebesgue."

Theorem 5.8. Lebesgue.

Let f be a bounded function on the closed, bounded interval $[a, b]$. Then f is Riemann integrable over $[a, b]$ if and only if the set of points in $[a, b]$ at which f fails to be continuous has measure zero.

Note. It is reasonable to refer to this as the "Riemann-Lebesgue Theorem," since it classifies the *Riemann* integrability of a function in terms of its continuity by referring to *Lebesgue* measure. In fact, Bernhard Riemann (1826–1866) knew this result himself, though he certainly would not express the fact in terms of measure. Quoting from page 265 of *A History of Analysis*, edited by Hans Niels Jahnke (*History of Mathematics*, Volume 24) American Mathematical Society (2003):

> "The 'total length of the intervals, in which the oscillations are $> \sigma$ [can thus. . .], whatever σ may be, be made arbitrarily small by an appropriate choice of d' (Riemann 1854, 273). The statement is also sufficient. Riemann established this, too (in an annotation, Dedekind, as editor of Riemann's works, referred to a handwritten note of Riemann's completing the proof). . . "

The reference here is:

Riemann, B. G. F. 1854. *Über die Darstellbarkeit einer Function durch eine trigonometrische Reihe.* Habalitationsschrift, Universität Góttingen, or in: *Gesammelte mathematische Werke*, 259–303.

The reference to "oscillations" is similar to the proof of the Riemann-Lebesgue Theorem given in "The Riemann-Lebesgue Theorem" handout. The idea that the "total length of the intervals. . . can be made arbitrarily small" is equivalent to the idea of measure zero (which, as you now know, can be addressed without all the concerns of a full blown theory of measure).

Note. The story of the Riemann-Lebesgue Theorem carries on several more decades. Quoting from Morris Kline's *Mathematical Through from Ancient to Modern Times* (Oxford University Press, 1972): "Darboux [in *Ann. de l'Ecole Norm. Sup.*, (2), 4, 1875, 57–112] then shows that a bounded function will be integrable on $[a, b]$ if and only if the discontinuities in $f(x)$ constitute a set of measure zero. By the latter he meant that the points of discontinuity can be enclosed in a finite set of intervals whose total length is arbitrarily small. This formulation of the integrability condition was given by a number of other men in the same year (1875). [page 960]" Lebesgue himself seems to have presented a proof (and he is the first to really have the concept of measure clearly defined). "In fact Lebesgue showed (*Le cons sur l'integration et la recherche des fonctions primitives*, 1904) that a bounded function is Riemann integrable if and only if the points of discontinuity form a set of measure 0. [page 1046]"

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