Section 6.2. Differentiability of Monotone Functions

Note. In this section we prove that a monotone function on an open interval (bounded or unbounded) is a.e. differentiable on the interval.

Note. Royden and Fitzpatrick adopt the unconventional terminology that a singleton $\{a\}$ is a *degenerate interval*; that is, $\{a\} = [a, b]$ where $b = a$.

Definition. A collection $\mathcal{F}$ of closed, bounded, nondegenerate intervals is said to cover a set E *in the sense of Vitali* provided for each point $x \in E$ and $\varepsilon > 0$, there is an interval $I \in \mathcal{F}$ that contains x and has $\ell(I) < \varepsilon$.

Example. Let $E = [0, 1]$ and $\mathcal{I}_1 = \{(x - \varepsilon/2, x + \varepsilon/2 \mid x \in E, 0 < \varepsilon < 1\}$. Then $\mathcal{I}_1$ is a covering in the sense of Vitali of E. Let

$$\mathcal{I}_2 = \{(x - \varepsilon/2.x + \varepsilon/2) \mid x \in E \cap \mathbb{Q}, \varepsilon \in (0, 1] \cap \mathbb{Q}\}.$$

Then $\mathcal{I}_2$ is a countable cover of E in the sense of Vitali. Is there a finite $\mathcal{I}_3$ that is a cover of E in the sense of Vilati? (No!)

The Vitali Covering Lemma.

Let E be a set of finite outer measure and $\mathcal{F}$ a collection of closed, bounded intervals that covers E in the sense of Vitali. Then for each $\varepsilon > 0$, there is a finite disjoint subcollection $\{I_k\}_{k=1}^n$ of $\mathcal{F}$ for which

$$m^* \left[E \setminus \bigcup_{k=1}^{n} I_k \right] < \varepsilon. \tag{2}$$

Definition. For a real-valued function f and an interior point x of its domain, the *upper derivative* of f at x, $\overline{D}[f(x)]$ and the *lower derivative* of f at x, $\underline{D}[f(x)]$ are:

$$\overline{D}[f(x)] = \lim_{h\to 0} \left[\sup_{0<|t|\le h} \frac{f(x+t)-f(x)}{t} \right];$$

$$\underline{D}[f(x)] = \lim_{h\to 0} \left[\inf_{0<|t|\le h} \frac{f(x+t)-f(x)}{t} \right].$$

We say that f is *differentiable* at x if $\overline{D}[f(x)] = \underline{D}[f(x)]$ and denote this common value as $f'(x)$.

Note. The following lemma is a generalization of the Mean Value Theorem. If f is continuous on $[c,d]$ and differentiable on (c,d) with $f'(x) \ge \alpha$ for $x \in (c,d)$, then $\alpha(d-c) \le [f(d)-f(c)]$.

Lemma 6.3. Let f be an increasing function on the closed, bounded interval $[a, b]$. Then for each $\alpha > 0$,

$$m^*\{x \in (a, b) \mid \overline{D}[f(x)] \geq \alpha\} \leq \frac{1}{\alpha}[f(b) - f(a)] \tag{7}$$

and

$$m^*\{x \in (a, b) \mid \overline{D}[f(x)] = \infty\} = 0. \tag{8}$$

Note. The following result is our most "modern" result concerning monotone functions.

Lebesgue Theorem.

If the function f is monotone on the open interval (a, b), then it is differentiable almost everywhere on (a, b).

Note. The converse of Lebesgue's Theorem holds in the following sense. For any set E of measure zero a subset of (a, b), there exists an increasing function on (a, b) that is not differentiable on E. This is Problem 6.10 (though a careful reading of the problem indicates that f is not differentiable at each point of E—the possibility of other points of nondifferentiability is not disallowed...at least, not in Problem 6.10).

Note. Royden and Fitzpatrick quote Frigyes Riesz and Béla Szőkefnalvy-Nagy (in their *Functional Analysis*, Mineola, NY: Dover Publishing, 1990; this is an unabridged republication of the 1955 version of their book) as saying that Lebesgue's Theorem is "one of the most striking and most important in real variable theory." Lebesgue's Theorem was published in 1904 (*Leçons sur l'intégration et la recherche des fonctions primitives*, page 128) and its consequences "helped restore confidence in the harmony of mathematic[al] analysis" (Royden and Fitzpatrick, page 113).

Definition. Let f be integrable over the closed, bounded interval $[a, b]$. Extend f to take the value $f(b)$ on $(b, b+1]$. For $0 < h \le 1$, define the *divided difference function* $\mathrm{Diff}_h[f]$ and *average value function* $\mathrm{Av}_h[f]$ of $[a, b]$ by

$$\mathrm{Diff}_h[f(x)] = \frac{f(x+h) - f(x)}{h} \text{ and } \mathrm{Av}_h[f(x)] = \frac{1}{h}\int_x^{x+h} f \text{ for all } x \in [a, b].$$

Corollary 6.4. Let f be an increasing function on the closed, bounded interval $[a, b]$. Then f' is integrable over $[a, b]$ and

$$\int_a^b f' \le f(b) - f(a).$$

Note. We only used the fact that f is increasing on $[a, b]$ to get the bound in Corollary 6.4. We can improve the bound under the same hypotheses to conclude that

$$\int_a^b f' \le \sup_{x \in (a,b)} f(x) - \inf_{x \in (a,b)} f(x).$$

Note. For the Cantor-Lebesgue function ϕ of Section 2.7, we have that ϕ is increasing and $\phi' = 0$ on the complement of the Cantor set with respect to $[0, 1]$. So $\phi' = 0$ a.e. on $[0, 1]$ and hence $\int_{[0,1]} \phi' = 0$. But $\phi(0) = 0$ and $\phi(1) = 1$, so function ϕ shows that we may get a strict inequality in Corollary 6.4.

Note. Of course, we want to weaken the hypotheses of Corollary 6.4 AND tighten the result to get a Fundamental Theorem of "Lebesgue" Calculus. In Section 6.4 we introduce absolute continuity and accomplish a version of a fundamental theorem.

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