Section 6.3. Functions of Bounded Variation: Jordan's Theorem

Note. In this section we define functions of bounded variation and show that such a function is the difference of two increasing functions (this is Jordan's Theorem).

Note. Let f be a real-valued function on the closed, bounded interval $[a, b]$ and let $P = \{x_0, x_1, \ldots, x_k\}$ be a partition of $[a, b]$. Define the *variation* of f with respect to P as

$$V(f, P) = \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})|.$$

The *total variation* of f on $[a, b]$ is

$$TV(f) = \sup\{V(f, P) \mid P \text{ is a partition of } [a, b]\}.$$

Definition. A real-valued function f on the closed, bounded interval $[a, b]$ is of *bounded variation* on $[a, b]$ if $TV(f) < \infty$.

Note. The image of a function $f : \mathbb{R} \to \mathbb{C}$ of bounded variation on $[a, b]$ is the type of path we integrate over in the complex setting. For details on the relevant definitions in this setting, the development of the Riemann-Stieltjes integral, and the definition of a complex line ("path") integral, see my online notes for Complex Analysis 1 and 2 (MATH 5510/5520):

`http://faculty.etsu.edu/gardnerr/5510/notes/IV-1.pdf`.

Example. Let f be increasing on $[a, b]$. Then f is of bounded variation since $TV(f) = f(b) - f(a)$.

Example. Function f defined on set E is *Lipschitz* if there exists $c \geq 0$ such that $|f(x') - f(x)| \leq c|x' - x|$ for all $x', x \in E$. A differentiable function is Lipschitz, and a Lipschitz function is continuous. So Lipschitz is a condition between continuity and differentiability. If f is Lipschitz on $[a, b]$ then f is of bounded variation on $[a, b]$ and $TV(f) \leq c(b - a)$. For further discussion of Lipschitz functions, see my "Primer on Lipschitz functions" online at:

`http://faculty.etsu.edu/gardnerr/5510/CSPACE.pdf`.

Example. Define the function f on $[0, 1]$ by

$$f(x) = \begin{cases} x\cos(\pi/2x) & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0. \end{cases}$$

Then f is continuous on $[0, 1]$, but f is not of bounded variation on $[0, 1]$. For $n \in \mathbb{N}$, and partition $P_n = \{0, 1/(2n), 1/(2n - 1), \ldots, 1/3, 1/2, 1\}$ of $[0, 1]$ we have

$$V(f, P_n) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

Since this is the partial sum for a harmonic series, then $TV(f) = \infty$.

Lemma 6.5. Let the function f be of bounded variation on the closed, bounded interval $[a, b]$. Then f has the following explicit expression as the difference of two increasing functions on $[a, b]$:

$$f(x) = [f(x) + TV(f_{[a,x]})] - TV(f_{[a,x]}) \text{ for all } x \in [a, b].$$

Jordan's Theorem.

A function f is of bounded variation on the closed, bounded interval $[a, b]$ if and only if it is the difference of two increasing functions on $[a, b]$. When f is written as such a difference, it is called a *Jordan decomposition* of f.

Corollary 6.6. If the function f is of bounded variation on the closed, bounded interval $[a, b]$, then it is differentiable almost everywhere on the open interval (a, b) and f' is integrable over $[a, b]$.

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