Section 6.4. Absolutely Continuous Functions

Note. In this section we define absolute continuity. Our Fundamental Theorem of Calculus will hold for this type of function.

Definition. A real-valued function f on a closed, bounded interval $[a, b]$ is *absolutely continuous* on $[a, b]$ if for each $\varepsilon > 0$ there is $\delta > 0$ such that for every finite disjoint collection $\{(a_k, b_k)\}_{k=1}^n$ of open intervals in (a, b),

$$\text{if } \sum_{k=1}^{n}(b_k - a_k) < \delta \text{ then } \sum_{k=1}^{n}|f(b_k) - f(a_k)| < \varepsilon.$$

Note. If f is absolutely continuous on $[a, b]$ then we see that, with $n = 1$, f is uniformly continuous on $[a, b]$. The converse does not hold, though.

Example. The Cantor-Lebesgue function ϕ from Section 2.7 is increasing and continuous on $[0, 1]$. However, ϕ is not absolutely continuous. Let $n \in \mathbb{N}$. At the nth state fo the construction of the Cantor set, a disjoint collection $\{[c_k, d_k]\}_{k=1}^{2^n}$ of 2^n subintervals of $[0, 1]$ have been constructed that cover the Cantor set, each of which has length $1/3^n$. The Cantor-Lebesgue function is constant on each of the open intervals that comprise the complement in $[0, 1]$ of this collection of intervals. Therefore, since ϕ is increasing and $\phi(1) - \phi(0) = 1 - 0 = 1$,

$$\sum_{k=1}^{2^n}(d_k - c_k) = 2^n(1/3^n) = (2/3)^n \text{ while } \sum_{k=1}^{2^n}(\phi(d_k) - \phi(c_k) = 1 - 0 = 1$$

(since $c_0 = 0$ and $d_{2^n} = 1$; the $[c_k, d_k]$ cover the Cantor set and are subintervals of $[0,1]$). So with $0 < \varepsilon \leq 1$, there is no sufficiently small $\delta > 0$ to give the condition of absolute continuity (choosing n sufficiently large, we can get $(2/3)^n < \delta$, but the sum of function values are still equal to 1).

Note. Linear combinations of absolutely continuous functions are absolutely continuous. The composition of absolutely continuous functions may not be absolutely continuous (see Problem 6.43).

Note. We now give three results relating absolute continuity to some other properties of functions.

Proposition 6.7. If the function f is Lipschitz on a closed, bounded interval $[a, b]$, then it is absolutely continuous on $[a, b]$.

Note. The converse of Proposition 6.7 does not hold. $f(x) = \sqrt{x}$ is absolutely continuous on $[0,1]$, but not Lipschitz on $[0,1]$ (see problem 6.37).

Theorem 6.8. Let the function f be absolutely continuous on the closed, bounded interval $[a, b]$. Then f is the difference of increasing absolutely continuous functions and, in particular, is of bounded variation.

Theorem 6.9. Let the function f be continuous on the closed, bounded interval $[a, b]$. Then f is absolutely continuous on $[a, b]$ if and only if the family of divided difference functions $\{\text{Diff}_h[f]\}_{0<h\leq 1}$ is uniformly integrable over $[a, b]$.

Note. For a nondegenerate closed, bounded interval $[a, b]$, let $\mathcal{F}_{\text{Lip}}$, $\mathcal{F}_{\text{AC}}$, and $\mathcal{F}_{\text{BV}}$ denote the families of functions on $[a, b]$ which are, respectively, Lipschitz, absolutely continuous, and of bounded variation. By Proposition 6.7 and Theorem 6.8 we have $\mathcal{F}_{\text{Lip}} \subseteq \mathcal{F}_{\text{AC}} \subseteq \mathcal{F}_{\text{BV}}$. In fact, each of these families are closed under linear combinations.

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