Section 6.5. Integrating Derivatives: Differentiating Indefinite Integrals

Note. Recall the Fundamental Theorem of Calculus as stated in *Thomas' Calculus* 12th Edition (see my online notes at `http://faculty.etsu.edu/gardnerr/1910/Notes-12E/c5s4.pdf`):

The Fundamental Theorem of Calculus, Part 1. If f is continuous on $[a, b]$ then the function

$$F(x) = \int_a^x f(t)\,dt$$

has a derivative at every point x in $[a, b]$ and

$$\frac{dF}{dx} = \frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x).$$

The Fundamental Theorem of Calculus, Part 2. If f is continuous at every point of $[a, b]$ and if F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Our Fundamental Theorem also has two parts, though we prove "the second part" first (in Calculus 1, the proof of the second part depends on the first part).

Theorem 6.10. Fundamental Theorem of Lebesgue Calculus, Part 2. Let the function f be absolutely continuous on the closed, bounded interval $[a, b]$. Then f is differentiable almost everywhere on (a, b), its derivative f' is integrable over $[a, b]$, and

$$\int_a^b f' = f(b) - f(a).$$

Proof. By the change of variables formula (14) on page 113 and in the proof of Corollary 6.4, we have

$$\int_u^v \mathrm{Diff}_h[f] = \mathrm{Av}_x[f(v)] = \mathrm{Av}_h[f(u)]$$

for all $a \le u < v \le b$. Since f is absolutely continuous then it is continuous and so

$$\lim_{h \to 0^+} (\mathrm{Av}_h[f(u)]) \lim_{h \to 0^+} \left(\frac{1}{h} \int_u^{u+h} f \right) = f(u)$$

(as shown in detail in Calculus 1). So replacing h with $1/n$ and letting $n \to 0$ we have (with $u = a$ and $v = b$)

$$\left(\int_a^b \mathrm{Diff}_{1/n}[f] \right) = f(b) - f(a). \qquad (29)$$

Theorem 6.8 implies that f is the difference of two increasing functions on $[a, b]$ and therefore, by Lebesgue's Theorem, f is differentiable a.e. on (a, b). Since the sequence $\{\mathrm{Diff}_{1/n}[f]\}$ converges to f' for the x values where $f'(x)$ exists, then $\{\mathrm{Diff}_{1/n}[f]\}$ converges pointwise a.e. on (a, b) to f'. By Theorem 6.9, since f is absolutely continuous, the family $\{\mathrm{Diff}_{1/n}[f]\}_{n=1}^{\infty} \subseteq \{\mathrm{Diff}_h[f]\}_{0<h\le 1}$ is uniformly integrable over $[a, b]$. By the Vitali Convergence Theorem,

$$\lim_{n \to \infty} \left(\int_a^b \mathrm{Diff}_{1/n}[f] \right) = \int_a^b \lim_{n \to \infty} (\mathrm{Diff}_{1/n}[f]) = \int_a^b f'.$$

Combining this with (29) gives the result. ∎

Note. Our Fundamental Theorem differs from that of Calculus 1 in that we are integrating f' where we assume that f itself is absolutely continuous. In Calculus 1 it is assumed that the *integrand* is continuous. Our integrand f' may not be continuous—in fact, it may not even exist on a set of measure zero. For example, consider

$$f(x) = \begin{cases} x & \text{if } x \in [0,1] \\ 1 & \text{if } x \in (1,2]. \end{cases}$$

Then $f(x)$ is absolutely continuous on $[0,2]$. Also,

$$f'(x) = \begin{cases} 1 & \text{if } x \in (0,1) \\ 0 & \text{if } x \in (1,2), \end{cases}$$

so f is differentiable a.e. on $(0,2)$, but f' is not continuous on $[0,2]$. So the Calculus 1 Fundamental Theorem does not apply (although the techniques of Calculus 1 *can* be applied to evaluate $\int_0^2 f'(x)\,dx$). By our Fundamental Theorem,

$$\int_0^2 f' = f(2) - f(0) = 1.$$

Note. For the first part of our Fundamental Theorem we need the idea of an antiderivative. Recall from Calculus 1 that the indefinite integral of f is the *set* of all antiderivatives of f. We take a slightly different approach here.

Definition. A function f on closed, bounded interval $[a,b]$ is the *indefinite integral* of g over $[a,b]$ if g is Lebesgue integrable over $[a,b]$ and

$$f(x) = f(a) + \int_a^x g \text{ for all } x \in [a,b].$$

Note. The following result gives a nice classification of absolutely continuous functions.

Theorem 6.11. A function f on a closed, bounded interval $[a, b]$ is absolutely continuous on $[a, b]$ if and only if it is an indefinite integral over $[a, b]$.

Corollary 6.12. Let the function f be monotone on the closed, bounded interval $[a, b]$. Then f is absolutely continuous on $[a, b]$ if and only if

$$\int_a^b f' = f(b) - f(a).$$

Note. The following technical result is used in the proof of the other half of our Fundamental Theorem.

Lemma 6.13. Let f be integrable over the closed bounded interval $[a, b]$. Then $f(x) = 0$ for almost all $x \in [a, b]$ if and only if $\int_{x_1}^{x_2} f = 0$ for all $(x_1, x_2) \subseteq [a, b]$.

Note. Now for our Fundamental Theorem of Lebesgue Calculus Part 1! As expected, it states that the derivative of an integral yields the integrand.

Theorem 6.14. Fundamental Theorem of Lebesgue Calculus, Part 1. Let f be integrable over the closed, bounded interval $[a,b]$. Then

$$\frac{d}{dx}\left[\int_a^x f\right] = f(x) \text{ for almost all } x \in (a,b).$$

Proof. Define the function F on $[a,b]$ by $F(x) = \int_a^x f$ for all $x \in [a,b]$. Theorem 6.11 tells us, since F is an indefinite integral, that it is absolutely continuous. Therefore, by Theorem 6.10, F is differentiable a.e. on (a,b) and its derivative F' is integrable. By Lemma 6.13, to show that the integrable function $F' - f$ vanishes a.e. on $[a,b]$ it suffices to show that its integral over every closed subinterval of $[a,b]$. By Theorem 6.10, with $[a,b]$ replaced by $[x_1,x_2]$ we have

$$\begin{aligned}\int_{x_1}^{x_2}[F'-f] &= \int_{x_1}^{x_2} F' - \int_{x_1}^{x_2} f \text{ by linearity}\\ &= F(x_2) - F(x_1) - \int_{x_1}^{x_2} f \text{ by definition of } F\\ &= \int_{x_1}^{x_2} f - \int_{x_1}^{x_2} f \text{ by additivity}\\ &= 0.\end{aligned}$$

So by Lemma 6.13, $F' - f = 0$ for almost all $x \in [a,b]$; or $F' = f$ for almost all $x \in (a,b)$. ∎

Definition. A function of bounded variation on set E is *singular* if its derivative is 0 a.e. on E.

Note. The Cantor-Lebesgue function is an increasing nonconstant singular function on $[0,1]$.

Note. Let f be of bounded variation on $[a, b]$. By Corollary 6.6, f' exists a.e. on $[a, b]$ and f' is integrable on $[a, b]$. Define

$$g(x) = \int_a^x f' \text{ and } h(x) = f(x) - \int_a^x f' \text{ for all } x \in [a, b].$$

Then $f = g + h$ on $[a, b]$. By Theorem 6.11, g is absolutely continuous since it is an indefinite integral. By the Fundamental Theorem of Lebesgue Calculus Part 1,

$$\begin{aligned} \frac{d}{dx}[h(x)] &= \frac{d}{dx}\left[f(x) - \int_a^x f'\right] \\ &= f'(x) - f'(x) \text{ for almost all } x \in (a, b) \\ &= 0, \end{aligned}$$

and so $h'(x) = 0$ for almost all $x \in (a, b)$. That is, h is singular on $[a, b]$. We have dcomposed f into the sum of an absolutely continuous function plus a singular function under the assumption that f is of bounded variation.

Definition. For bounded variation function f on $[a, b]$, if $f = g + h$ where g is of bounded variation and absolutely continuous and h is of bounded variation and singular, the sum $f = g + h$ is a *Lebesgue decomposition* of f.

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