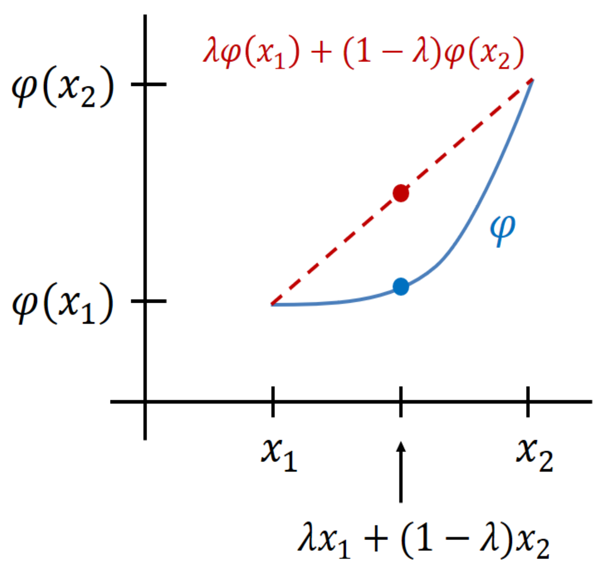
Section 6.6. Convex Functions

Note. In this section we present several classical results concerning "convex functions" (examples of which are the concave up functions from Calculus 1). We also prove a result (Jensen's Inequality) which concerns convex functions and integrable functions. Throughout this section, "(a,b)" denotes an open interval that may be either bounded or unbounded.

Definition. A real-valued function φ on (a,b) is *convex* provided for each pair of points $x_1, x_2 \in (a,b)$ and each λ with $0 \le \lambda \le 1$ we have

$$\varphi(\lambda x_1 + (1-\lambda)x_2) \le \lambda\varphi(x_1) + (1-\lambda)\varphi(x_2).$$

Note. As λ ranges over $[0,1]$, the values of $\lambda x_1 + (1-\lambda)x_2$ ranges from x_2 to x_1 and the values of $\lambda\varphi(x_1) + (1-\lambda)\varphi(x_2)$ range ("linearly") from $\varphi(x_2)$ to $\varphi(x_1)$. So the geometric interpretation is that the *chord* joining $(x_1, \varphi(x_1))$ to $(x_2, \varphi(x_2))$ lies above the corresponding function values (as does a "concave up" function).



Note. We can also describe convex functions in terms of slopes of chords. With $x_1 < x_2$ in (a,b) and $x \in (x_1,x_2)$, we have $x = \lambda x_1 + (1-\lambda)x_2$ for $\lambda = (x_2 - x)/(x_2 - x_1)$. So from the definition of convex we have

$$\varphi(x) \leq \left(\frac{x_2 - x}{x_2 - x_1}\right)\varphi(x_1) + \left(\frac{x - x_1}{x_2 - x_1}\right)\varphi(x_2)$$

since $1 - \lambda = 1 - (x_2 - x)/(x_2 - x_1) = (x - x_1)/(x_2 - x_1)$. This gives

$$(x_2 - x_1)\varphi(x) \leq (x_2 - x)\varphi(x_1) + (x - x_1)\varphi(x_2) \qquad (38')$$

which holds if and only if

$$x_2\varphi(x) - (x_2 - x)\varphi(x_1) \leq (x - x_1)\varphi(x_2) + x_1\varphi(x)$$

if and only if

$$x_2\varphi(x) - x\varphi(x) - (x_2 - x)\varphi(x_1) \leq (x - x_1)\varphi(x_2) + x_1\varphi(x) - x\varphi(x)$$

if and only if

$$(x_2 - x)(\varphi(x) - \varphi(x_1)) \leq (x - x_1)(\varphi(x_2) - \varphi(x))$$

if and only if

$$\frac{\varphi(x) - \varphi(x_1)}{x - x_1} \leq \frac{\varphi(x_2) - \varphi(x)}{x_2 - x} \qquad (39)$$

for all $x_1 < x < x_2$ in (a,b). So geometrically in terms of slopes of chords, convexity of φ on (a,b) implies that the slope of the chord from $(x_1, \varphi(x_1))$ to $(x, \varphi(x))$ is less than or equal to the slope of the chord from $(x, \varphi(x))$ to $(x_2, \varphi(x_2))$.

Note. The following result confirms that "concave up" functions of Calculus 1 are, in fact, convex.

Proposition 6.15. If φ is differentiable on (a,b) and its derivative φ' is increasing, then φ is convex. In particular, if φ'' exists on (a,b) and $\varphi'' \geq 0$ on (a,b), then φ is convex.

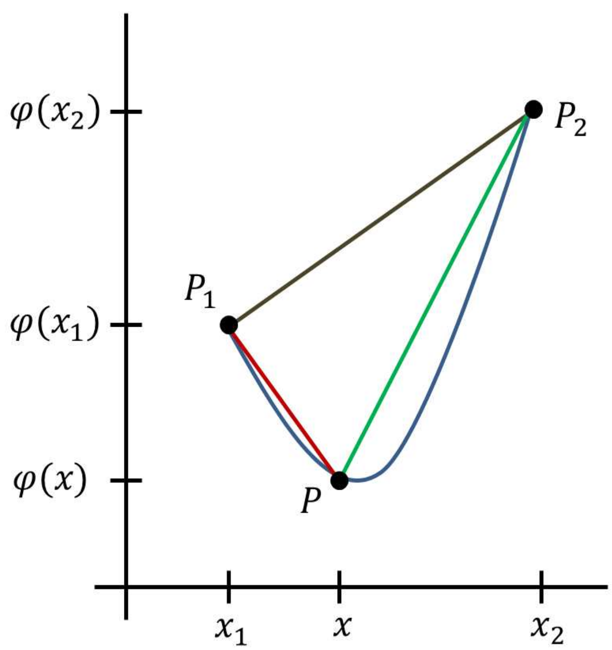
Example. Function $\varphi(x) = x^p$ is convex on $(0,\infty)$ for $p \geq 1$ since $\varphi''(x) = p(p-1)x^{p-2} > 0$ for $x \in (0,\infty)$. We'll use the convexity of this function in our exploration of L^p spaces where $p \geq 1$.

The Chordal Slope Lemma. Let φ be convex on (a,b). If $x_1 < x < x_2$ are in (a,b), then for points $P_1 = (x_1, \varphi(x_1))$, $P = (x, \varphi(x))$, and $P_2 = (x_2, \varphi(x_2))$ we have

$$\frac{\varphi(x) - \varphi(x_1)}{x - x_1} \leq \frac{\varphi(x_2) - \varphi(x_1)}{x_2 - x_1} \leq \frac{\varphi(x_2) - \varphi(x)}{x_2 - x}.$$

That is, the slope of $\overline{P_1P}$ is less than or equal to the slope of $\overline{P_1P_2}$, which is less than or equal to the slope of $\overline{PP_2}$.

Note. The picture for the Chordal Slope Lemma for a concave up function is:



Definition. For a function g on interval (a, b) and point $x_0 \in (a, b)$, if

$$\lim_{h \to 0,\, h<0} \frac{g(x_0 + h) - g(x_0)}{h} \text{ exists and is finite}$$

then it is the *left-hand derivative* of g at x_0, denoted $g'(x_0^-)$. The *right-hand derivative* denoted $g'(x_0^+)$ is similarly defined by restricting $h > 0$.

Lemma 6.16. Let φ be a convex function on (a, b). Then φ has left-hand and right-hand derivative at each point $x \in (a, b)$. Moreover, for points $u < v$ in (a, b) these one-sided derivatives satisfy the following inequality:

$$\varphi'(u^-) \leq \varphi'(u^+) \leq \frac{\varphi(v) - \varphi(u)}{v - u} \leq \varphi'(v^-) \leq \varphi(v^+).$$

Note. Lemma 6.16 implies that for $u < v$ in (a, b), if $\varphi'(u)$ and $\varphi'(v)$ exist, then $\varphi'(u) \leq \varphi'(v)$. The proof is left as homework.

Corollary 6.17. Let φ be a convex function on (a, b). Then φ is Lipschitz, and therefore absolutely continuous, on each closed, bounded subinterval $[c, d]$ of (a, b).

Theorem 6.18. Let φ be a convex function on (a, b). Then φ is differentiable except at a countable number of points and its derivative φ' is an increasing function.

Definition. Let φ be a convex function on (a, b) and let x_0 belong to (a, b). For real number m, the line $y = m(x - x_0) + \varphi(x_0)$ (which passes through the point $(x_0, \varphi(x_0))$ is a *supporting line* at x_0 for the graph of φ if this line always lies below the graph of φ. That is, if

$$\varphi(x) \geq m(x - x_0) + \varphi(x_0) \text{ for all } x \in (a, b).$$

Lemma. Let φ be a convex function on (a, b) and let x_0 belong to (a, b). Then there is a supporting line at x_0 for the graph of φ for every slope between $\varphi'(x_0^-)$ and $\varphi'(x_0^+)$.

Note. A convex function on an interval is a.e. differentiable by Theorem 6.18, so a convex function on an interval is Riemann integrable on the interval (that is, the Riemann integral exists) so we use the notation for Riemann integrals in the following result.

Jensen's Inequality. Let φ be a convex function on $(-\infty, \infty)$, f an integrable function over $[0, 1]$, and $\varphi \circ f$ also integrable over $[0, 1]$. Then

$$\varphi\left(\int_0^1 f(x)\,dx\right) \leq \int_0^1 (\varphi \circ f)(x)\,dx.$$

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