Chapter 7. The L^p Spaces: Completeness and Approximation

Section 7.1. Normed Linear Spaces

Note. We assume E is a measurable set. Denote the set of all measurable extended real-valued functions on E that are finite a.e. on E as $\mathcal{F}$.

Definition. Define an equivalence relation $\cong$ on $\mathcal{F}$ as $f \cong g$ if and only if $f = g$ a.e. on E.

Note. $\cong$ is reflexive, symmetric, and transitive and so $\cong$ really is an equivalence relation. Therefore $\cong$ partitions $\mathcal{F}$ into equivalence classes. We denote the equivalence class containing f as $[f]$. Since f is finite a.e., there is an element of $[f]$ which is finite on all of E. We can define for all $\alpha, \beta \in \mathbb{R}$ and all $[f]$ and $[g]$, a linear combination $\alpha[f] + \beta[g]$ which is the equivalence class containing $\alpha f_F + \beta g_F$ where $f_F \in [f]$, $g_F \in [g]$ and f_F, g_F are finite on E.

Definition. Let X be a set of functions (or equivalence classes of functions). Then X is a *linear space* if for all $f, g \in X$ (or $[f], [g] \in X$) and $\alpha, \beta \in \mathbb{R}$, $\alpha f + \beta g \in X$ (or $\alpha[f] + \beta[g] \in X$).

Example 7.1.A. Let $1 \le p < \infty$. Define $L^p(E)$ to be the set of equivalence classes of functions for which $\int_E |f|^p < \infty$. Notice that for $a, b \in \mathbb{R}$,

$$|a+b| \le |a| + |b| \le 2\max\{|a|,|b|\}$$

and so

$$|a+b|^p \le 2^p \left(\max\{|a|,|b|\}\right)^p \le 2^p(|a|^p + |b|^p).$$

So if $f, g \in L^p(E)$, then

$$|f+g|^p \le 2^p(|f|^p + |g|^p) \text{ on } E$$

and

$$\begin{aligned} \int_E |f+g|^p &\le \int_E 2^p(|f|^p + |g|^p) \text{ by monotonicity of the integral} \\ &= 2^p\left(\int_E |f|^p + \int_E |g|^p\right) \text{ by linearity} \\ &< \infty \end{aligned}$$

and so $f + g \in L^p(E)$. Also, $f \in L^p(E)$ implies that $\alpha f \in L^p(E)$ for all $\alpha \in \mathbb{R}$. So $L^p(E)$ is a linear space.

Definition. For $f \in \mathcal{F}$, we say f is *essentially bounded* if for some $M \ge 0$ (called an *essential upper bound*) for which $|f(x)| \le M$ for almost all $x \in E$. Define $L^\infty(E)$ to be the set of equivalence classes of the essentially bounded functions on E.

Note. "Clearly" $L^\infty(E)$ is a linear space.

Note. For our study of Banach and Hilbert spaces, we will no longer distinguish between a *function* f and the *equivalence class* $[f]$.

Note. We will see parallels between the L^p spaces $(1 \leq p \leq \infty)$ and vector spaces.

Definition. Let X be a linear space. A real-valued functional (i.e., a function with X as its domain and $\mathbb{R}$ as its codomain) $\|\cdot\|$ on X is a *norm* if for all $f, g \in X$ and for all $\alpha \in \mathbb{R}$:

(1) $\|f + g\| \leq \|f\| + \|g\|$ (Triangle Inequality).

(2) $\|\alpha f\| = |\alpha|\|f\|$ (Positive Homogeneity).

(3) $\|f\| \geq 0$ and $\|f\| = 0$ if and only if $f = 0$ (Nonnegativity).

Definition. A *normed linear space* is a linear space X with a norm $\|\cdot\|$.

Example 7.1.B. Define $\|\cdot\|_1$ on $L^1(E)$ as $\|f\|_1 = \int_E |f|$. To verify $\|\cdot\|_1$ is a norm, notice that for $f, g \in L^1(E)$, f and g are finite a.e. on E (or WLOG everywhere on E) and so $|f + g| \leq |f| + |g|$ a.e. on E. So by monotonicity and linearity of integration

$$\|f + g\|_1 = \int_E |f + g| \leq \int_E (|f| + |g|) = \int_E |f| + \int_E |g| = \|f\|_1 + \|g\|_1,$$

and therefore $\|\cdot\|_1$ satisfies the Triangle Inequality. Next, for $\alpha \in \mathbb{R}$

$$\|\alpha f\|_1 = \int_E |\alpha f| = \int_E |\alpha||f| = |\alpha| \int_E |f| = |\alpha|\|f\|_1,$$

and Positive Homogeneity holds. Finally,

$$\|f\|_1 = \int_E |f| = 0 \text{ if and only if } f = 0 \text{ a.e. on } E$$

by Proposition 4.9, and so nonnegativity clearly holds. □

Example 7.1.C. Define $\|\cdot\|_\infty$ on $L^\infty(E)$ as

$$\begin{aligned} \|f\|_\infty &= \inf\{M \mid M \text{ is an essential upper bound of } f \text{ on } E\} \\ &= \inf\{M \mid |f| \leq M \text{ a.e. on } E\}. \end{aligned}$$

$\|f\|_\infty$ is called the *essential supremum* of f. It can be shown that $\|\cdot\|_\infty$ is a norm (see page 138) and hence $L^\infty(E)$ is a normed linear space.

Example. Let $[a,b] \subset \mathbb{R}$. Denote the linear space of continuous real-valued functions on $[a,b]$ as $C[a,b]$. Define for $f \in C[a,b]$

$$\|f\|_{\max} = \max_{x \in [a,b]} |f(x)|.$$

It can be shown that $\|\cdot\|_{\max}$ is a norm (called the *maximum norm*) and so $C[a,b]$ is a normed linear space (Problem 7.1).

Example. Let ℓ^1 be the set of all absolutely summable sequences of real numbers:

$$\ell^1 = \left\{ (a_1, a_2, \ldots) \,\middle|\, \sum_{k=1}^{\infty} |a_k| < \infty \right\}.$$

Define $\|\cdot\|_1$ on ℓ^1 as

$$\|\{a_k\}\|_1 = \sum_{k=1}^{\infty} |a_k|.$$

Then $\|\cdot\|_1$ is a norm and ℓ^1 is a normed linear space (Problem 7.5a).

Example. Let ℓ^∞ be the set of all bounded sequences of real numbers. Define $\|\cdot\|_\infty$ as

$$\|\{a_k\}\|_\infty = \sup\{|a_k|\}.$$

Then $\|\cdot\|_\infty$ is a norm on ℓ^∞ and ℓ^∞ is a normed linear space (Problem 7.5b).

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