Chapter 7. The $L^p$ Spaces: Completeness and Approximation

Section 7.1. Normed Linear Spaces

Note. We assume $E$ is a measurable set. Denote the set of all measurable extended real-valued functions on $E$ that are finite a.e. on $E$ as $\mathcal{F}$.

Definition. Define an equivalence relation $\sim$ on $\mathcal{F}$ as $f \sim g$ if and only if $f = g$ a.e. on $E$.

Note. $\sim$ is reflexive, symmetric, and transitive and so $\sim$ really is an equivalence relation. Therefore $\sim$ partitions $\mathcal{F}$ into equivalence classes. We denote the equivalence class containing $f$ as $[f]$. Since $f$ is finite a.e., there is an element of $[f]$ which is finite on all of $E$. We can define for all $\alpha, \beta \in \mathbb{R}$ and all $[f]$ and $[g]$, a linear combination $\alpha[f] + \beta[g]$ which is the equivalence class containing $\alpha f_F + \beta g_F$ where $f_F \in [f]$, $g_F \in [g]$ and $f_F, g_F$ are finite in $E$.

Definition. Let $X$ be a set of functions (or equivalence classes of functions). Then $X$ is a linear space if for all $f, g \in X$ (or $[f], [g] \in X$) and $\alpha, \beta \in \mathbb{R}$, $\alpha f + \beta g \in X$ (or $\alpha[f] + \beta[g] \in X$).
Example. Let $1 \leq p < \infty$. Define $L^p(E)$ to be the set of equivalence classes of functions for which $\int_E |f|^p < \infty$. Notice that for $a, b \in \mathbb{R}$,

$$|a + b| \leq |a| + |b| \leq 2 \max\{|a|, |b|\}$$

and so

$$|a + b|^p \leq 2^p (\max\{|a|, |b|\})^p \leq 2^p (|a|^p + |b|^p) .$$

So if $f, g \in L^p(E)$, then

$$|f + g|^p \leq 2^p (|f|^p + |g|^p) \text{ on } E$$

and

$$\int_E |f + g|^p \leq \int_E 2^p (|f|^p + |g|^p) \text{ by monotonicity of the integral}$$

$$= 2^p \left( \int_E |f|^p + \int_E |g|^p \right) \text{ by linearity}$$

$$< \infty$$

and so $f + g \in L^p(E)$. Also, $f \in L^p(E)$ implies that $\alpha f \in L^p(E)$ for all $\alpha \in \mathbb{R}$. So $L^p(E)$ is a linear space.

Definition. For $f \in \mathcal{F}$, we say $f$ is essentially bounded if for some $M \geq 0$ (called an essential upper bound) for which $|f(x)| \leq M$ for almost all $x \in E$. Define $L^\infty(E)$ to be the set of equivalence classes of the essentially bounded functions on $E$.

Note. “Clearly” $L^\infty(E)$ is a linear space.
Note. For our study of Banach and Hilbert spaces, we will no longer distinguish between a function $f$ and the equivalence class $[f]$.

Note. We will see parallels between the $L^p$ spaces ($1 \leq p \leq \infty$) and vector spaces.

**Definition.** Let $X$ be a linear space. A real-valued functional (i.e., a function with $X$ as its domain and $\mathbb{R}$ as its codomain) $\| \cdot \|$ on $X$ is a **norm** if for all $f, g \in X$ and for all $\alpha \in \mathbb{R}$:

(1) $\| f + g \| \leq \| f \| + \| g \|$ (Triangle Inequality).

(2) $\| \alpha f \| = |\alpha| \| f \|$ (Positive Homogeneity).

(3) $\| f \| \geq 0$ and $\| f \| = 0$ if and only if $f = 0$ (Nonnegativity).

**Definition.** A **normed linear space** is a linear space $X$ with a norm $\| \cdot \|$.

**Example.** Define $\| \cdot \|_1$ on $L^1(E)$ as $\| f \|_1 = \int_E |f|$. To verify $\| \cdot \|_1$ is a norm, notice that for $f, g \in L^1(E)$, $f$ and $g$ are finite a.e. on $E$ (or WLOG everywhere on $E$) and so $|f + g| \leq |f| + |g|$ a.e. on $E$. So by monotonicity and linearity of integration

$$\| f + g \|_1 = \int_E |f + g| \leq \int_E (|f| + |g|) = \int_E |f| + \int_E |g| = \| f \|_1 + \| g \|_1,$$

and therefore $\| \cdot \|_1$ satisfies the Triangle Inequality. Next, for $\alpha \in \mathbb{R}$

$$\| \alpha f \|_1 = \int_E |\alpha f| = \int_E |\alpha||f| = |\alpha| \int_E |f| = |\alpha| \| f \|_1,$$
and Positive Homogeneity holds. Finally,
\[ \|f\|_1 = \int_E |f| = 0 \text{ if and only if } f = 0 \text{ a.e. on } E \]
by Proposition 4.9, and so nonnegativity clearly holds.

\[ \Box \]

**Example.** Define \( \| \cdot \|_\infty \) on \( L^\infty(E) \) as
\[
\|f\|_\infty = \inf \{ M \mid M \text{ is an essential upper bound of } f \text{ on } E \} = \inf \{ M \mid |f| \leq M \text{ a.e. on } E \}.
\]

\( \|f\|_\infty \) is called the *essential supremum* of \( f \). It can be shown that \( \| \cdot \|_\infty \) is a norm (see page 138) and hence \( L^\infty(E) \) is a normed linear space.

**Example.** Let \([a, b] \subset \mathbb{R}\). Denote the linear space of continuous real-valued functions on \([a, b]\) as \( C[a, b] \). Define for \( f \in C[a, b] \)
\[
\|f\|_{\max} = \max_{x \in [a, b]} |f(x)|.
\]
It can be shown that \( \| \cdot \|_{\max} \) is a norm (called the *maximum norm*) and so \( C[a, b] \) is a normed linear space (Problem 7.1).
Example. Let $\ell^1$ be the set of all absolutely summable sequences of real numbers:

$$\ell^1 = \left\{ (a_1, a_2, \ldots) \left| \sum_{k=1}^{\infty} |a_k| < \infty \right. \right\}.$$  

Define $\| \cdot \|_1$ on $\ell^1$ as

$$\|\{a_k\}\|_1 = \sum_{k=1}^{\infty} |a_k|.$$  

Then $\| \cdot \|_1$ is a norm and $\ell^1$ is a normed linear space (Problem 7.5a).

Example. Let $\ell^\infty$ be the set of all bounded sequences of real numbers. Define $\| \cdot \|_\infty$ as

$$\|\{a_k\}\|_\infty = \text{sup}\{|a_k|\}.$$  

Then $\| \cdot \|_\infty$ is a norm on $\ell^\infty$ and $\ell^\infty$ is a normed linear space (Problem 7.5b).  

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