Section 7.2. The Inequalities of Young, Hölder, and Minkowski

Note. We now put a norm on $L^p(E)$ for $1 < p < \infty$... of course, we'll need to confirm that it is a norm.

Definition. For measurable E, $1 < p < \infty$ and $f \in L^p(E)$, define

$$\|f\|_p = \left\{\int_E |f|^p\right\}^{1/p}.$$

Note. $\|\cdot\|_p$ satisfies Positive homogeneity:

$$\|\alpha f\|_p = \left\{\int_E |\alpha f|^p\right\}^{1/p} = |\alpha|\left\{\int_E |f|^p\right\}^{1/p} = |\alpha|\|f\|_p.$$

$\|\cdot\|_p$ satisfies Nonnegativity: Since $|f| \geq 0$, then

$$\|f\|_p = \left\{\int_E |f|^p\right\}^{1/p} \geq 0$$

and by Proposition 4.9,

$$\|f\|_p = \left\{\int_E |f|^p\right\}^{1/p} = 0 \text{ if and only if } |f| = 0 \text{ a.e. on } E,$$

that is, if and only if $f \in [0]$ (where $[0]$ is the equivalence class containing the zero function). The Triangle Inequality is more difficult to establish and that's the purpose of this section.

Definition. The *conjugate* of $p \in (1, \infty)$ is the number $q = \dfrac{p}{p-1}$. (Notice that $\frac{1}{p} + \frac{1}{q} = 1$.) 1 and ∞ are conjugates of each other.

Note. We'll see an intimate relationship between the spaces L^p and L^q.

Young's Inequality.

For $1 < p < \infty$ and q the conjugate of p, for any positive a and b,

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Theorem 7.1. Hölder's Inequality. Let E be a measurable set, $1 \leq p < \infty$, and q the conjugate of p. If $f \in L^p(E)$ and $g \in L^q(E)$, then fg is integrable over E and

$$\int_E |fg| \leq \|f\|_p \|g\|_q.$$

This is *Hölder's Inequality*. Moreover, if $f \neq 0$, then the function

$$f^* = \begin{cases} \|f\|_p^{1-p} \mathrm{sgn}(f)|f|^{p-1} & \text{if } p > 1 \\ \mathrm{sgn}(f) & \text{if } p = 1 \end{cases}$$

is an element of $L^q(E)$,

$$\int_E f f^* = \|f\|_p \tag{4}$$

and $\|f^*\|_q = 1$.

Definition. For $f \in L^p(E)$, $f \neq 0$, the function f^* defined in Theorem 7.1 is the *conjugate function* of f.

Note. We will use Hölder's Inequality to prove the Triangle Inequality in $L^p(E)$. In $\mathbb{R}^n$, we use the Cauchy-Schwarz Inequality (or simply the "Schwarz Inequality"), which states that for all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ we have $|\mathbf{v} \cdot \mathbf{w}| \leq \|\mathbf{v}\| \|\mathbf{w}\|$, to prove the triangle inequality as follows:

$$
\begin{aligned}
\|\mathbf{v} + \mathbf{w}\|^2 &= (\mathbf{v} + \mathbf{w}) \cdot (\mathbf{v} + \mathbf{w}) \\
&= \mathbf{v} \cdot \mathbf{v} + 2\mathbf{v} \cdot \mathbf{w} + \mathbf{w} \cdot \mathbf{w} \\
&= \|\mathbf{v}\|^2 + 2\mathbf{v} \cdot \mathbf{w} + \|\mathbf{w}\|^2 \\
&\leq \|\mathbf{v}\|^2 + 2\|\mathbf{v}\|\|\mathbf{w}\| + \|\mathbf{w}\|^2 \text{ by the Cauchy-Schwarz Inequality} \\
&= (\|\mathbf{v}\| + \|\mathbf{w}\|)^2
\end{aligned}
$$

So there should be a parallel between Hölder's Inequality and the Cauchy-Schwarz Inequality. This can be seen when we take $p = q = 2$, in which case Hölder implies $\int_E |fg| \leq \|f\|_2 \|g\|_2$ (Royden and Fitzpatrick state this as the Cauchy-Schwarz Inequality on page 142). The right hand side of this inequality has norms as we would expect, and the left hand side consists of an integral *similar* to the idea of a dot product in $\mathbb{R}^n$. As we'll see, the idea of a dot product (or "inner product") is only valid in $L^2(E)$.

Corollary 7.2. Let E be a measurable set and $1 < p < \infty$. Suppose $\mathcal{F}$ is a family of functions in $L^p(E)$ that is bounded in $L^p(E)$ in the sense that there is a constant M for which $\|f\|_p \leq M$ for all $f \in \mathcal{F}$. Then the family is uniformly integrable on E.

Minkowski's Inequality.

Let E be measurable and $1 \leq p \leq \infty$. If f and g belong to $L^p(E)$, then $f + g \in L^p(E)$ and

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

Note. Minkowski's Inequality both (1) proves that L^p is closed under addition (and hence is a linear space), and (2) proves that $\|\cdot\|_p$ is a norm. Therefore $L^p(E)$ is a normed linear space. In fact, when $m(E) < \infty$, the $L^p(E)$ spaces are nested.

Corollary 7.3. Let E be measurable, $m(E) < \infty$, and $1 \leq p_1 < p_2 \leq \infty$. Then $L^{p_2}(E) \subset L^{p_1}(E)$. Furthermore, $\|f\|_{p_1} \leq c\|f\|_{p_2}$ for all $f \in L^{p_2}(E)$ where $c = (m(E))^{(p_2-p_1)/(p_1p_2)}$ if $p_2 < \infty$ and $c = (m(E))^{1/p_1}$ if $p_2 = \infty$.

Note. When $m(E) < \infty$, the inclusion in Corollary 7.3 is strict. For example, with $E = (0, 1]$ and α such that $-1/p_1 < \alpha < -1/p_2$. Define $f(x) = x^\alpha$. Then $f \in L^{p_1}(E)$ and $f \notin L^{p_2}(E)$ (where, as above, $1 \leq p_1 < p_2 < \infty$). This is confirmed in Problem 7a.

Note. If $m(E) = \infty$, there is no inclusion relationship between the L^p spaces. For example, with $E = (0, \infty)$ and $f(x) = \dfrac{x^{-1/2}}{1 + |\ln x|}$, then $f \in L^p(0, \infty)$ if and only if $p = 2$. This is supposed to be Problem 7b, but there is a typographical error in the book.

Note. To this point, we have not established a relationship between the L^p spaces for $1 \leq p < \infty$. As is often the case, "∞" plays the role of a limit. That is the case here, as is to be demonstrated in Problem 7.18:

Problem 7.18. Assume $m(E) < \infty$. For $f \in L^\infty(E)$,

$$\lim_{p \to \infty} \|f\|_p = \|f\|_\infty.$$

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