Section 7.3. $L^p$ is Complete: The Riesz-Fischer Theorem

Note. In this section, we introduce the equipment to show that $L^p$ is complete for $1 \leq p \leq \infty$. Therefore these spaces are complete normed linear spaces (such a space is called a Banach space).

Definition. A sequence $\{f_n\}$ in a linear space $X$ with norm $\| \cdot \|$ is said to converge to $f \in X$ if $\lim_{n \to \infty} \| f - f_n \| = 0$, denoted $\{f_n\} \to f$ or $\lim_{n \to \infty} f_n = f$.

Definition. A sequence $\{f_n\}$ in a normed linear space is said to be Cauchy if for all $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $\| f_n - f_m \| < \varepsilon$ for all $m, n \geq N$.

Definition. A normed linear space $X$ is said to be complete if every Cauchy sequence in $X$ converges to an element of $X$. A complete normed linear space is a Banach space.

Example. $\mathbb{R}$ under absolute value is a Banach space. $\mathbb{R}^n$ is a Banach space under the Euclidean norm. The function space $C[a, b]$ under the max norm is a Banach space (Problem 7.31). $L^\infty(E)$ is a Banach space (Problem 7.33).
Proposition 7.4. Let $X$ be a normed linear space. Then every convergent sequence in $X$ is Cauchy. Moreover, a Cauchy sequence in $X$ converges if it has a convergent subsequence.

Definition. A sequence $\{f_n\}$ in a normed linear space is rapidly Cauchy provided there is a convergent series of positive numbers $\sum_{k=1}^{\infty} \varepsilon_k$ for which $\|f_{k+1} - f_k\| \leq \varepsilon_k^2$ for all $k$.

Notice. A sequence may be Cauchy but not rapidly Cauchy. Consider the normed linear space $\mathbb{R}$ with norm of absolute value. Let

$$\{f_k\} = \left\{ \sum_{i=1}^{k} \frac{1}{i^2} \right\}.$$

Then $\{f_n\}$ consists of partial sums of the $p$-series with $p = 2$, and so $\{f_n\}$ converges (to $\pi^2/6$) and so is Cauchy. However,

$$|f_{k+1} - f_k| = \left| \sum_{i=1}^{k+1} \frac{1}{i^2} - \sum_{i=1}^{k} \frac{1}{i^2} \right| = \frac{1}{(k+1)^2} = \varepsilon_k^2,$$

and so $\varepsilon_k$ must be $1/(k+1)$. But then $\sum_{k=1}^{\infty} \varepsilon_k = \sum_{k=1}^{\infty} \frac{1}{k+1}$ diverges (Harmonic series) and so $\{f_k\}$ is not rapidly Cauchy (this is Problem 7.23).

Note. However, rapidly Cauchy sequences are Cauchy:

Proposition 7.5. Let $X$ be a normed linear space. Then every rapidly Cauchy sequence in $X$ is Cauchy. Furthermore every Cauchy sequence has a rapidly Cauchy subsequence.
Note. We need the following, which has a lengthy proof, for the proof that the $L^p$ spaces are complete.

**Theorem 7.6.** Let $E$ be measurable and $1 \leq p \leq \infty$. Then every rapidly Cauchy sequence in $L^p(E)$ converges both with respect to the $L^p$ norm and pointwise a.e. on $E$ to a function in $L^p(E)$.

Note. The following allows us to conclude that $L^p(E)$ is a Banach space for $1 \leq p \leq \infty$.

**The Riesz-Fischer Theorem.** Let $E$ be measurable and $1 \leq p \leq \infty$. Then $L^p(E)$ is a Banach space. Moreover, if $\{f_n\} \to f$ in $L^p$ then there is a subsequence of $\{f_n\}$ which converges pointwise a.e. on $E$ to $f$.

Note. The Riesz-Fischer Theorem implies that $L^p$-convergence implies pointwise a.e. convergence of a subsequence. Let $E = [0, 1], 1 \leq p < \infty$ and $f_n = n^{1/p} \chi_{(0,1/n]}$. Then $f_n \to 0$ pointwise but $\|f_n - 0\|_p = 1$ and so $f_n$ does not converge to 0 with respect to the $L^p$ norm.

Note. The following result gives necessary and sufficient conditions for pointwise convergence to imply $L^p$ convergence.
Theorem 7.7. Let $E$ be measurable and $1 \leq p < \infty$. Suppose $\{f_n\}$ is a sequence in $L^p(E)$ that converges pointwise a.e. on $E$ to $f \in L^p(E)$. Then $\{f_n\} \to f$ with respect to the $L^p$ norm if and only if

$$
\lim_{n \to \infty} \int_E |f_n|^p = \int_E |f|^p,
$$

that is $\|f_n\|_p \to \|f\|_p$.

Note. You have been exposed to the $\ell^p$ sequence spaces, $1 \leq p \leq \infty$, in the homework. These spaces are also examples of Banach spaces. Another example is the space

$$
\mathbb{R}^\omega = \{ x = (x_1, x_2, \ldots) \mid x_i \in \mathbb{R} \text{ for } i \in \mathbb{N} \}.
$$

$\mathbb{R}^\omega$ is a linear space where for any $a, b \in \mathbb{R}$ and $x, y \in \mathbb{R}^\omega$ we define $ax + by = z$ with $z_i = ax_i + by_i$ for all $i \in \mathbb{N}$. It is shown in our Introduction to Topology (MATH 4357/5357) class that

$$
D(x, y) = \sup_{i \in \mathbb{N}} \{ \overline{d}(x_i, y_i)/i \},
$$

where $\overline{d}(a, b) = \min\{|a - b|, a\}$, is a metric on $\mathbb{R}^\omega$ and so a norm on $\mathbb{R}^\omega$ is

$$
\|x\| = D(x, 0) = \sup_{i \in \mathbb{N}} \{ \overline{d}(x_i, 0)/i \} = \sup_{i \in \mathbb{N}} \{ \min\{|x_i|, 1\}/i \} = \sup_{i \in \mathbb{N}} \{ \min\{|x_i|/i, 1/i\} \}.
$$

So $\mathbb{R}^\omega$ is a normed linear space (notice that $\|x\| \leq 1$ for all $x \in \mathbb{R}^\omega$). In fact, $\mathbb{R}^\omega$ is complete with respect to the metric, and so $\mathbb{R}^\omega$ is an example of Banach space. See my online notes for Introduction to Topology at http://faculty.etsu.edu/gardnerr/5357/notes.htm (see sections 20 and 43 for details).

Revised 2/12/2019