Section 7.3. L^p is Complete: The Riesz-Fischer Theorem

Note. In this section, we introduce the equipment to show that L^p is complete for $1 \leq p \leq \infty$. Therefore these spaces are complete normed linear spaces (such a space is called a Banach space).

Definition. A sequence $\{f_n\}$ in a linear space X with norm $\|\cdot\|$ is said to *converge to* $f \in X$ if $\lim_{n\to\infty} \|f - f_n\| = 0$, denoted $\{f_n\} \to f$ or $\lim_{n\to\infty} f_n = f$.

Definition. A sequence $\{f_n\}$ in a normed linear space is said to be *Cauchy* if for all $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $\|f_n - f_m\| < \varepsilon$ for all $m, n \geq N$.

Definition. A normed linear space X is said to be *complete* if every Cauchy sequence in X converges to an element of X. A complete normed linear space is a *Banach space*.

Note. In the setting of $\mathbb{R}$ (an ordered field), completeness is dealt with in terms of a upper bounds and least upper bounds. The Axiom of completeness states that every set of real numbers with an upper bound has a least upper bound. It is this axiom that *makes* the real numbers a continuum (though you may have been introduced to the idea in terms of Dedekind cuts, which are equivalent to the Axiom of completeness). More details are in my online notes for Analysis 1 (MATH 4217/5217) on Section 1.3. The Completeness Axiom.

Note. In L^p we do not have an ordering so there is no "upper" or "least." So we approach completeness in terms of Cauchy sequences, which only require the idea of distance (which we have, since we have a norm on L^p for each $1 \leq p \leq \infty$). We know that a sequence of real numbers is Cauchy if and only if it converges (see my Analysis 1 notes on Section 2.3. Bolzano-Weierstrass Theorem). Think of it as: A Cauchy sequence *wants* to converge! ... and a Cauchy sequence *will* converge, unless there is a hole in the space at the "point" to which the Cauchy sequence wants to converge. For example, a sequence of real number (no terms of which is $\sqrt{2}$) which converges to $\sqrt{2}$ will NOT converge in the space $\mathbb{R} \setminus \{\sqrt{2}\}$, since there is a hole in the space at the point to which the sequence want to converge. We also use Cauchy sequences to define completeness in the complex field (which also has no ordering). See my online notes for Complex Analysis 1 (MATH 5510) on Section II.3. Sequences and Completeness, where completeness is defined in terms of Cauchy sequences in a metric space in general, and used to show that $\mathbb{C}$ is complete (see Proposition II.3.6 in the complex notes).

Example. $\mathbb{R}$ under absolute value is a Banach space. $\mathbb{R}^n$ is a Banach space under the Euclidean norm. The function space $C[a, b]$ under the max norm is a Banach space (Problem 7.31). $L^\infty(E)$ is a Banach space (Problem 7.33).

Proposition 7.4. Let X be a normed linear space. Then every convergent sequence in X is Cauchy. Moreover, a Cauchy sequence in X converges if it has a convergent subsequence.

Definition. A sequence $\{f_n\}$ in a normed linear space is *rapidly Cauchy* provided there is a convergent series of positive numbers $\sum_{k=1}^{\infty} \varepsilon_k$ for which $\|f_{k+1} - f_k\| \le \varepsilon_k^2$ for all k.

Note 7.3.A. A sequence may be Cauchy but not rapidly Cauchy. Consider the normed linear space $\mathbb{R}$ with norm of absolute value. Let

$$\{f_k\} = \left\{ \sum_{i=1}^{k} \frac{1}{i^2} \right\}.$$

Then $\{f_n\}$ consists of partial sums of the p-series with $p = 2$, and so $\{f_n\}$ converges (to $\pi^2/6$) and so is Cauchy. However,

$$|f_{k+1} - f_k| = \left| \sum_{i=1}^{k+1} \frac{1}{i^2} - \sum_{i=1}^{k} \frac{1}{i^2} \right| = \frac{1}{(k+1)^2} = \varepsilon_k^2$$

and so ε_k must be $1/(k+1)$. But then $\sum_{k=1}^{\infty} \varepsilon_k = \sum_{k=1}^{\infty} \frac{1}{k+1}$ diverges (Harmonic series) and so $\{f_k\}$ is not rapidly Cauchy (this is Problem 7.23).

Note. However, rapidly Cauchy sequences are Cauchy:

Proposition 7.5. Let X be a normed linear space. Then every rapidly Cauchy sequence in X is Cauchy. Furthermore every Cauchy sequence has a rapidly Cauchy subsequence.

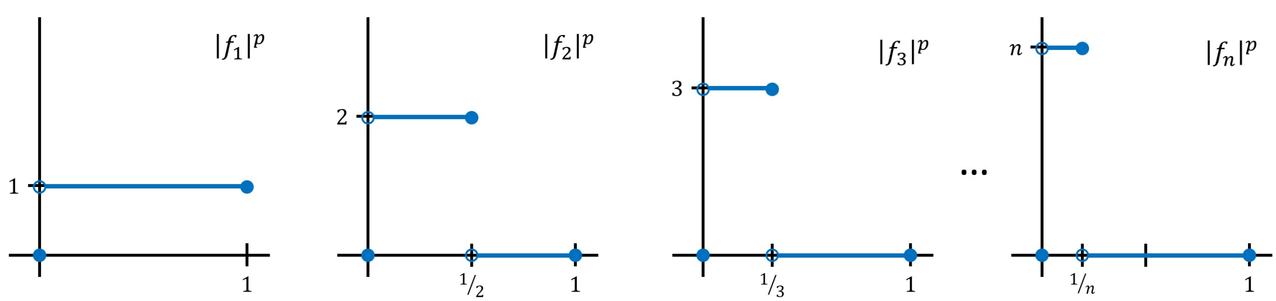
Note. We need the next result for the proof that the L^p spaces are complete.

Theorem 7.6. Let E be measurable and $1 \leq p \leq \infty$. Then every rapidly Cauchy sequence in $L^p(E)$ converges both with respect to the L^p norm and pointwise a.e. on E to a function in $L^p(E)$.

Note. The following allows us to conclude that $L^p(E)$ is a Banach space for $1 \leq p \leq \infty$.

The Riesz-Fischer Theorem. Let E be measurable and $1 \leq p \leq \infty$. Then $L^p(E)$ is a Banach space. Moreover, if $\{f_n\} \to f$ in L^p then there is a subsequence of $\{f_n\}$ which converges pointwise a.e. on E to f.

Note. The Riesz-Fischer Theorem implies that L^p-convergence implies pointwise a.e. convergence of a subsequence. However, the converse does not hold. Let $E = [0,1]$, $1 \leq p < \infty$ and $f_n = n^{1/p}\chi_{(0,1/n]}$. Then $f_n \to 0$ pointwise but $\|f_n - 0\|_p = 1$ and so f_n does not converge to 0 with respect to the L^p norm.



Note. The next theorem gives necessary and sufficient conditions for pointwise convergence of a sequence of functions in L^p to imply convergence with respect to the L^p norm in L^p (that is, for pointwise convergence to imply L^p convergence).

Theorem 7.7. Let E be measurable and $1 \leq p < \infty$. Suppose $\{f_n\}$ is a sequence in $L^p(E)$ that converges pointwise a.e. on E to $f \in L^p(E)$. Then $\{f_n\} \to f$ with respect to the L^p norm if and only if

$$\lim_{n\to\infty} \int_E |f_n|^p = \int_E |f|^p,$$

that is $\|f_n\|_p \to \|f\|_p$.

Note. You have been exposed to the ℓ^p *sequence spaces*, $1 \leq p \leq \infty$, in the homework. These spaces are also examples of Banach spaces. Another example is the space

$$\mathbb{R}^\omega = \{\mathbf{x} = (x_1, x_2, \ldots) \mid x_i \in \mathbb{R} \text{ for } i \in \mathbb{N}\}.$$

$\mathbb{R}^\omega$ is a linear space where for any $a, b \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^\omega$ we define $a\mathbf{x} + b\mathbf{y} = \mathbf{z}$ with $z_i = ax_i + by_i$ for all $i \in \mathbb{N}$. It is shown in our Introduction to Topology (MATH 4357/5357) class that

$$D(\mathbf{x}, \mathbf{y}) = \sup_{i\in\mathbb{N}}\{\overline{d}(x_i, y_i)/i\},$$

where $\overline{d}(a, b) = \min\{|a - b|, a\}$, is a metric on $\mathbb{R}^\omega$ and so a norm on $\mathbb{R}^\omega$ is

$$\|\mathbf{x}\| = D(\mathbf{x}, \mathbf{0}) = \sup_{i\in\mathbb{N}}\{\overline{d}(x_i, 0)/i\} = \sup_{i\in\mathbb{N}}\{\min\{|x_i|, 1\}/i\} = \sup_{i\in\mathbb{N}}\{\min\{|x_i|/i, 1/i\}\}.$$

So $\mathbb{R}^\omega$ is a normed linear space (notice that $\|\mathbf{x}\| \leq 1$ for all $\mathbf{x} \in \mathbb{R}^\omega$). In fact, $\mathbb{R}^\omega$ is complete with respect to the metric, and so $\mathbb{R}^\omega$ is an example of Banach space.

See my online notes for Introduction to Topology (MATH 4357/5357) on Section 20. The Metric Topology and Section 43. Complete Metric Spaces.

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