Section 7.4. Approximation and Separability

Note. We want to approximate elements of $L^p(E)$ with more “familiar” functions, such as continuous functions.

**Definition.** Let $X$ be a normed linear space with norm $\| \cdot \|$. Given two subsets $\mathcal{F}$ and $\mathcal{G}$ of $X$ with $\mathcal{F} \subset \mathcal{G}$, we say that $\mathcal{F}$ is **dense** in $\mathcal{G}$, if for each function $g \in \mathcal{G}$ and $\varepsilon > 0$, there is a function $f \in \mathcal{F}$ for which $\|f - g\| < \varepsilon$.

**Note.** If $\mathcal{F}$ is dense in $\mathcal{G}$, then for any $g \in \mathcal{G}$ there is a sequence $\{f_n\} \subset \mathcal{F}$ such that $f_n \to g$ with respect to the norm by Exercise 7.36.

**Proposition 7.9.** Let $E$ be measurable and $1 \leq p \leq \infty$. Then the subspace of simple functions in $L^p(E)$ is dense in $L^p(E)$.

**Note.** The following is a related but “simpler” result.

**Proposition 7.10.** Let $[a, b]$ be a closed, bounded interval and $1 \leq p < \infty$. Then the subspace of step functions on $[a, b]$ is dense in $L^p([a, b])$.

**Definition.** A normed linear space $X$ is **separable** if there is a countable subset that is dense in $X$. 

Note. We’ll see the idea of separability again when we deal with Hilbert spaces. In fact, sometimes a Hilbert space is defined as a separable inner product space.

**Theorem 7.11.** Let $E$ be measurable and $1 \leq p < \infty$. Then $L^p(E)$ is separable.

Note. $L^\infty$ is not, in general, separable. Consider $L^\infty([a, b])$ and suppose there is a countable set $\{f_n\}_{n=1}^\infty$ that is dense in $L^\infty([a, b])$. For each $x \in [a, b]$, define $\eta(x) \in \mathbb{N}$ for which $\|\chi_{[a,x]} - f_{\eta(x)}\|_\infty < 1/2$ (which can be done since $\{f_n\}$ is dense in $L^\infty([a, b])$). For $x_1, x_2 \in [a, b]$ with $x_1 \neq x_2$, we have $\|\chi_{[a,x_1]} - \chi_{[a,x_2]}\|_\infty = 1$ and so $f_{\eta(x_1)} \neq f_{\eta(x_2)}$ by the Triangle Inequality and so $\eta(x_1) \neq \eta(x_2)$. That is $\eta$ is a one-to-one mapping from $[a, b]$ into $\mathbb{N}$. But this is impossible. Therefore, $\{f_n\}$ is not dense in $L^\infty([a, b])$ and $L^\infty([a, b])$ is not separable. This example is useful in the proof of Exercise 7.42.

**Theorem 7.12.** Let $E$ be measurable and $1 \leq p < \infty$. Then $C_c(E)$, the linear space of continuous functions on $E$ that vanish outside a bounded set, is dense in $L^p(E)$.

*Revised: 2/19/2019*