Section 7.4. Approximation and Separability

Note. We want to approximate elements of $L^p(E)$ with more "familiar" functions, such as continuous functions.

Definition. Let X be a normed linear space with norm $\|\cdot\|$. Given two subsets $\mathcal{F}$ and $\mathcal{G}$ of X with $\mathcal{F} \subset \mathcal{G}$, we say that $\mathcal{F}$ is *dense* in $\mathcal{G}$, if for each function $g \in \mathcal{G}$ and $\varepsilon > 0$, there is a function $f \in \mathcal{F}$ for which $\|f - g\| < \varepsilon$.

Note 7.4.A. Let $\mathcal{F}$ be a subset of a normed linear space X. By Exercise 7.36, $\mathcal{F}$ is dense in X if and only if each $g \in X$ is the limit of a sequence in $\mathcal{F}$.

Proposition 7.9. Let E be measurable and $1 \le p \le \infty$. Then the subspace of simple functions in $L^p(E)$ is dense in $L^p(E)$.

Note. The following is a related but "simpler" result.

Proposition 7.10. Let $[a, b]$ be a closed, bounded interval and $1 \le p < \infty$. Then the subspace of step functions on $[a, b]$ is dense in $L^p([a, b])$.

Definition. A normed linear space X is *separable* if there is a countable subset that is dense in X.

Note. We'll see the idea of separability again when we deal with Hilbert spaces. In fact, sometimes a Hilbert space is defined as a separable inner product space.

Theorem 7.11. Let E be measurable and $1 \leq p < \infty$. Then $L^p(E)$ is separable.

Note 7.4.B. L^∞ is not, in general, separable. Consider $L^\infty([a,b])$ and suppose there is a countable set $\{f_n\}_{n=1}^\infty$ that is dense in $L^\infty([a,b])$. For each $x \in [a,b]$, define $\eta(x) \in \mathbb{N}$ for which $\|\chi_{[a,x]} - f_{\eta(x)}\|_\infty < 1/2$ (which can be done since $\{f_n\}$ is dense in $L^\infty([a,b])$). For $x_1, x_2 \in [a,b]$ with $x_1 \neq x_2$, we have $\|\chi_{[a,x_1]} - \chi_{[a,x_2]}\|_\infty = 1$ and so $f_{\eta(x_1)} \neq f_{\eta(x_2)}$ by the Triangle Inequality and so $\eta(x_1) \neq \eta(x_2)$. That is η is a one-to-one mapping from $[a,b]$ into $\mathbb{N}$. But this is impossible. Therefore, $\{f_n\}$ is not dense in $L^\infty([a,b])$ and $L^\infty([a,b])$ is not separable. This example is useful in the proof of Exercise 7.42.

Theorem 7.12. Let E be measurable and $1 \leq p < \infty$. Then $C_c(E)$, the linear space of continuous functions on E that vanish outside a bounded set, is dense in $L^p(E)$.

Revised: 2/15/2023