Chapter 8. The $L^p$ Spaces: Duality and Weak Convergence

Section 8.1. The Riesz Representation

for the Dual of $L^p$, $1 \leq p < \infty$

Note. In this section, we introduce the idea of a linear functional on a linear space and find that the set of all bounded linear functionals on a given linear space is itself a linear space (called the dual space of the original space). The Riesz Representation Theorem classifies bounded linear functionals on $L^p(E)$ and allows us to show that the dual space of $L^p(E)$ is $L^q(E)$ where $\frac{1}{p} + \frac{1}{q} = 1$ and $1 \leq p < \infty$ (recall that such $p$ and $q$ are called conjugates).

Definition. A linear functional on a linear space $X$ is a real-valued function $T$ on $X$ such that for $g$ and $h$ in $X$ and $\alpha$ and $\beta$ real numbers, we have $T(\alpha g + \beta h) = \alpha T(g) + \beta T(h)$.

Note. For $E$ measurable, $1 \leq p < \infty$, $q$ the conjugate of $p$, and for $g \in L^q(E)$, define $T$ on $L^p(E)$ by $T(f) = \int_E g f$ for all $f \in L^p(E)$. By Hölder’s Inequality, $gf$ is integrable and so $T$ is defined. Since integration is linear, then $T$ is a linear functional. Also by Hölder’s Inequality, for all $f \in L^p(E)$, $|T(f)| \leq \|g\|_q \|f\|_p$. This will imply that $T$ is a “bounded linear functional” on $L^p$. The Riesz Representation Theorem states that every bounded linear functional on $L^p$ is of the form of $T$. 
**Definition.** For a normed linear space $X$, a linear functional $T$ on $X$ is said to be *bounded* if there is $M \geq 0$ for which $|T(f)| \leq M\|f\|$ for all $f \in X$. The infimum of all such $M$ is called the *norm* of $T$, denoted $\|T\|_*:

$$\|T\|_* = \inf_{f \in X} \{M \mid |T(f)| \leq M\|f\|\}.$$ 

**Note.** The linear functional $T$ defined above as $T(f) = \int_E gf$ is bounded by $\|g\|_q$.

**Note.** For $T$ a bounded linear functional on $X$, for all $f, h \in X$ we have $|T(f) - T(h)| \leq \|T\|_*\|f - h\|$. So if $\{f_n\} \to f$ with respect to the norm $\|\cdot\|$, then $\{T(f_n)\} \to T(f)$ in $\mathbb{R}$.

**Note.** In Exercise 8.1, it is to be shown that

$$\|T\|_* = \sup\{|T(f)| \mid f \in X, \|f\| = 1\}.$$ 

If $\|f\| < 1$, then $\|f/\|f\|\| = 1$ and $|T(f)| = |T(\|f\|f/\|f\|)| = \|f\|T(f/\|f\|)\| \leq \|f\|\|T\|_* < \|T\|_*$. So we can also say

$$\|T\|_* = \sup\{|T(f)| \mid f \in X, \|f\| \leq 1\}.$$ 

**Proposition 8.1.** Let $X$ be a normed linear space. Then the collection of bounded linear functional on $X$ is a linear space with $\|\cdot\|_*$ as a norm. The normed linear space of bounded functionals is called the *dual space* of $X$, denoted $X^*$.

**Proof.** Problem 8.2. □
Proposition 8.2. Let $E$ be measurable, $1 \leq p < \infty$, $q$ the conjugate of $p$, and $g$ belong to $L^q(E)$. Define the functional $T$ on $L^p(E)$ by $T(f) = \int_E g f$ for all $f \in L^p(E)$. Then $T$ is a bounded linear functional on $L^p(E)$ and $\|T\|_* = \|g\|_q$.

Note. We will see that the converse of Proposition 8.2 also holds. That is, every bounded linear functional of $L^p$ is of the form of $T$.

Proposition 8.3. Let $T$ and $S$ be bounded linear functionals on a normed linear space $X$. If $T = S$ on a dense subset $X_0$ of $X$, then $T = S$ on $X$.

Lemma 8.4. Let $E$ be measurable and $1 \leq p < \infty$. Suppose $g$ is integrable over $E$ and there is $M > 0$ such that $|\int_E g f| \leq M\|f\|_p$ for every simple function $f \in L^p(E)$. Then $g \in L^q(E)$ where $q$ is the conjugate of $p$. Moreover, $\|g\|_q \leq M$.

Theorem 8.5. Let $1 \leq p < \infty$. Suppose $T$ is a bounded linear functional on $L^p([a, b])$. Then there is a function $g \in L^q([a, b])$, where $q$ is the conjugate of $p$, for which $T(f) = \int_{[a,b]} g f$ for all $f \in L^p([a, b])$.

Note. The Riesz Representation Theorem extends Theorem 8.5 from $[a, b]$ to general measurable set $E$. 
Riesz Representation Theorem.

Let $E$ be measurable, $1 \leq p < \infty$, and $q$ the conjugate of $p$. Then for each $g \in L^q(E)$, define the bounded linear functional $R_g$ on $L^p(E)$ by $R_g(f) = \int_E gf$ for all $f \in L^p(E)$. Then for each bounded linear functional $T$ on $L^p(E)$, there is a unique function $g \in L^q(E)$ for which $R_g = T$ and $\|T\|_* = \|g\|_p$.

Note. Proposition 8.1 and the Riesz Representation Theorem combine to show that the dual space of $L^p(E)$ is $L^q(E)$, where $\frac{1}{p} + \frac{1}{q} = 1$ for $1 \leq p < \infty$. Surprisingly, the dual space of $L^\infty(E)$ is not (in general, at least) $L^1(E)$. That is, there is a bounded linear functional on $L^\infty(E)$ (for the case $E = [a, b]$) that is not of the form $T(f) = \int_{[a,b]} gf$ where $g \in L^1(E)$. The dual space of $L^\infty(E)$ is given in the Kantorovich Representation Theorem (Theorem 19.7) in the general setting of a measure space.

Note. In the event that $p = q = 2$, we see that the space $L^2(E)$ is “self dual.” The space $L^2(E)$ is special in other ways—it is the only $L^p$ space on which an inner product can be defined and is an example of a Hilbert space.

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