Chapter 8. The $\mathbf{L}^p$ Spaces: Duality and Weak Convergence

Section 8.1. The Riesz Representation for the Dual of L^p, $1 \leq p < \infty$

Note. In this section, we introduce the idea of a linear functional on a linear space and find that the set of all bounded linear functionals on a given linear space is *itself* a linear space (called the *dual space* of the original space). The Riesz Representation Theorem classifies bounded linear functionals on $L^p(E)$ and allows us to show that the dual space of $L^p(E)$ is $L^q(E)$ where $\frac{1}{p} + \frac{1}{q} = 1$ and $1 \leq p < \infty$ (recall that such p and q are called *conjugates*).

Definition. A *linear functional* on a linear space X is a real-valued function T on X such that for g and h in X and α and β real numbers, we have $T(\alpha g + \beta h) = \alpha T(g) + \beta T(h)$.

Note. For E measurable, $1 \leq p < \infty$, q the conjugate of p, and for $g \in L^q(E)$, define T on $L^p(E)$ by $T(f) = \int_E gf$ for all $f \in L^p(E)$. By Hölder's Inequality, gf is integrable and so T is defined. Since integration is linear, then T is a linear functional. Also by Hölder's Inequality, for all $f \in L^p(E)$, $|T(f)| \leq \|g\|_q \|f\|_p$. This will imply that T is a "bounded linear functional" on L^p. The Riesz Representation Theorem states that every bounded linear functional on L^p is of the form of T.

Definition. For a normed linear space X, a linear functional T on X is said to be *bounded* if there is $M \geq 0$ for which $|T(f)| \leq M\|f\|$ for all $f \in X$. The infimum of all such M is called the *norm* of T, denoted $\|T\|_*$:

$$\|T\|_* = \inf_{f \in X}\{M \mid |T(f)| \leq M\|f\|\}.$$

Note. The linear functional T defined above as $T(f) = \int_E gf$ is bounded by $\|g\|_q$. Notice that, in general, $|T(f)| \leq \|T\|_*\|f\|$.

Note 8.1.A. For T a bounded linear functional on X, for all $f, h \in X$ we have $|T(f) - T(h)| = |T(f-h)| \leq \|T\|_*\|f-h\|$. So if $\{f_n\} \to f$ with respect to the norm $\|\cdot\|$, then $\{T(f_n)\} \to T(f)$ in $\mathbb{R}$. That is, every bounded linear functional is continuous (see Exercise 8.3(a)).

Note. In Exercise 8.1, it is to be shown that

$$\|T\|_* = \sup\{|T(f)| \mid f \in X, \|f\| = 1\}.$$

If $\|f\| < 1$, then $\|f/\|f\|\| = 1$ and $|T(f)| = |T(\|f\|f/\|f\|)| = \|f\||T(f/\|f\|)| \leq \|f\|\|T\|_* < \|T\|_*$. So we can also say

$$\|T\|_* = \sup\{|T(f)| \mid f \in X, \|f\| \leq 1\}.$$

Proposition 8.1. Let X be a normed linear space. Then the collection of bounded linear functionals on X is a linear space with $\|\cdot\|_*$ as a norm. The normed linear space of bounded functionals is called the *dual space* of X, denoted X^*.

Note. The proof of Proposition 8.1 is to be given in Problem 8.2.

Proposition 8.2. Let E be measurable, $1 \le p < \infty$, q the conjugate of p, and g belong to $L^q(E)$. Define the functional T on $L^p(E)$ by $T(f) = \int_E gf$ for all $f \in L^p(E)$. Then T is a bounded linear functional on $L^p(E)$ and $\|T\|_* = \|g\|_q$.

Note. We will see that the converse of Proposition 8.2 also holds. That is, every bounded linear functional of L^p is of the form of T.

Proposition 8.3. Let T and S be bounded linear functionals on a normed linear space X. If $T = S$ on a dense subset X_0 of X, then $T = S$ on X.

Lemma 8.4. Let E be measurable and $1 \le p < \infty$. Suppose g is integrable over E and there is $M > 0$ such that $|\int_E gf| \le M\|f\|_p$ for every simple function $f \in L^p(E)$. Then $g \in L^q(E)$ where q is the conjugate of p. Moreover, $\|g\|_q \le M$.

Theorem 8.5. Let $1 \le p < \infty$. Suppose T is a bounded linear functional on $L^p([a, b])$. Then there is a function $g \in L^q([a, b])$, where q is the conjugate of p, for which $T(f) = \int_{[a,b]} gf$ for all $f \in L^p([a, b])$.

Note. The Riesz Representation Theorem extends Theorem 8.5 from $[a, b]$ to general measurable set E.

Riesz Representation Theorem.

Let E be measurable, $1 \leq p < \infty$, and q the conjugate of p. Then for each $g \in L^q(E)$, define the bounded linear functional $\mathcal{R}_g$ on $L^p(E)$ by $\mathcal{R}_g(f) = \int_E gf$ for all $f \in L^p(E)$. Then for each bounded linear functional T on $L^p(E)$, there is a unique function $g \in L^q(E)$ for which $\mathcal{R}_g = T$ and $\|T\|_* = \|g\|_q$.

Note. Proposition 8.1 and the Riesz Representation Theorem combine to show that the dual space of $L^p(E)$ "is" $L^q(E)$, where $\frac{1}{p} + \frac{1}{q} = 1$ for $1 \leq p < \infty$. Technically, the dual space of $L^p(E)$ (the elements of which are bounded linear functionals) is *isomorphic* to $L^q(E)$ (the elements of which are functions f where $|f|^q$ is integrable on E). In this section we follow the approach of Promislow's *A First Course In Functional Analysis* (John Wiley & Sons Publications, 2008) and simply say the dual space of $L^p(E)$ "is" $L^q(E)$, where in fact we mean that there is an onto ("surjective") isometry from the dual space of $L^p(E)$ to $L^q(E)$. By isometry, we mean that the mapping preserves norms (which in turn implies that it is also a one to one mapping). See my online notes for Fundamentals of Functional Analysis (MATH 5740). In Section 19.2. The Riesz Representation Theorem for the Dual of $L^p(X, \mu)$, $1 \leq p < \infty$, we show that $(L^p(X, \mu))^*$ is "isometrically isomorphic" to $L^q(X, \mu)$ for $1 \leq p < \infty$; this is equivalent to the idea of an onto isometry here.

Note. Surprisingly, the dual space of $L^\infty(E)$ is not (in general, at least) $L^1(E)$. That is, there is a bounded linear functional on $L^\infty(E)$ (for the case $E = [a, b]$) that is not of the form $T(f) = \int_{[a,b]} gf$ where $g \in L^1(E)$. The dual space of $L^\infty(E)$ is given in the Kantorovich Representation Theorem (Theorem 19.7 in Section 19.3. The Kantorovich Representation Theorem for the Dual of $L^\infty(X, \mu)$) in the general setting of a measure space.

Note. In the event that $p = q = 2$, we see that the space $L^2(E)$ is "self dual." The space $L^2(E)$ is special in other ways—it is the only L^p space on which an inner product can be defined and is an example of a Hilbert space.

Revised: 2/25/2023