Section 8.2. Weak Sequential Convergence in L^p

Note. The Bolzano-Weierstrass Theorem states that an infinite set of real numbers has a limit point, and implies that every bounded sequence of real numbers has a convergent subsequence (these results also hold in $\mathbb{R}^n$). It is easily seen that this can be violated in ℓ^2 by considering the sequence

$$(1,0,0,\ldots),(0,1,0,\ldots),(0,0,1,0,\ldots),\ldots,(0,0,\ldots,0,1,0,\ldots),\ldots$$

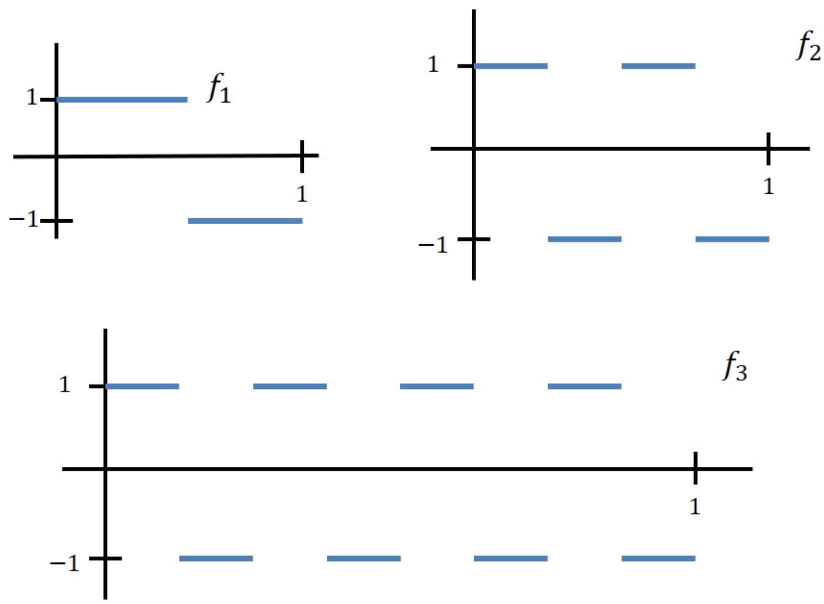
In fact, for any infinite dimensional normed linear space, there is a bounded sequence that has no convergent subsequence (this is Riesz's Theorem of Section 13.3. Compactness Lost: Infinite Dimensional Normed Linear Spaces). In this section, we introduce a new kind of convergence for sequences in L^p and give some necessary and sufficient conditions for a bounded sequence to converge in this new sense.

Example. Let $I = [0,1)$. For $n \in \mathbb{N}$, the sequence of *Radamacher functions* $\{f_n\}$ is defined as

$$f_x(x) = (-1)^k \text{ for } \frac{k}{2^n} \leq x < \frac{k+1}{2^n} \text{ where } 0 \leq k \leq 2^n - 1.$$

Notice that for $m \neq n$, $f_n \neq f_m$ on a set of measure $1/2$ (say $m < n$, then the intervals on which f_m is constant are each cut into 2^{n-m} pieces of the same size and f_n differs from f_m on half of these pieces). Fix $1 \leq p \leq \infty$. The $\{f_n\}$ is a bounded sequence in $L^p(I)$ since $\|f\|_p = 1$ for all $n \in \mathbb{N}$. For $n \neq m$, $|f_n - f_m| = 2$ on a set of measure $1/2$, and so

$$\|f_n - f_m\|_p = \left\{\int_I |f_n - f_m|^p\right\}^{1/p} = \left\{\frac{1}{2}2^p\right\}^{1/p} = 2^{(p-1)/p} = 2^{1-1/p}.$$



So no subsequence of $\{f_n\}$ is Cauchy in $L^p(I)$, and hence no subsequence of $\{f_n\}$ converges. So this is an example of a bounded sequence (in $L^p(I)$) with no convergent subsequence.

Definition. Let X be a normed linear space. A sequence $\{f_n\}$ in X is said to *converge weakly* in X to $f \in X$ provided $\lim_{n\to\infty} T(f_n) = T(f)$ for all $T \in X^*$. This is denoted $\{f_n\} \rightharpoonup f$ in X.

Note. Notice that the convergence of $\{f_n\}$ to f in X (still denoted "$\{f_n\} \to f$ in X") is stronger than weak convergence since it implies that for any $T \in X^*$ we have

$$\begin{aligned} |T(f_n) - T(f)| &= |T(f_n - f)| \text{ since } T \text{ is linear} \\ &\leq \|T\|_* \|f_n - f\| \text{ by the definition of } \|T\|_*, \end{aligned}$$

so if $\{f_n\} \to f$ in X (i.e., $\|f_n - f\| \to 0$) then $|T(f_n) - T(f)| \to 0$ (i.e., $T(f_n) = T(f)$). So convergence in X is sometimes called *strong convergence*.

Proposition 8.6. Let E be a measurable set, $1 \le p < \infty$, and q the conjugate of p (i.e., $\frac{1}{p} + \frac{1}{q} = 1$). Then $\{f_n\} \rightharpoonup f$ in $L^p(E)$ if and only if $\lim_{n\to\infty} \left(\int_E g f_n \right) = \int_E gf$ for all $g \in L^q(E)$.

Proof. This follows from the definition of weak convergence and the fact that by the Riesz Representation Theorem, $T \in X^*$ if and only if $T(f) = \int_E gf$ for some $g \in L^q(E)$. ∎

Lemma A. The limit of a weakly convergent sequence in $L^p(E)$ is unique, $1 \le p < \infty$.

Theorem 8.7. Let f be a measurable set and $1 \le p < \infty$. Suppose $\{f_n\} \rightharpoonup f$ in $L^p(E)$. Then $\{f_n\}$ is bounded in $L^p(E)$ and $\|f\|_p \le \liminf \|f_n\|_p$.

Note. The proof of Theorem 8.7 is long and requires Problem 8.18.

Note. The following is a generalization of Proposition 8.6.

Corollary 8.8. Let E be a measurable set, $a \le p < \infty$, and q the conjugate of p. Suppose $\{f_n\}$ converges weakly to f in $L^p(E)$ ($\{f_n\} \rightharpoonup f$ in $L^p(E)$) and $\{g_n\}$ converges strongly to g in $L^q(E)$ ($\{g_n\} \to g$ in $L^1(E)$). Then $\lim_{n\to\infty} \left(\int_E g_n f_n \right) = \int_E gf$.

Definition. Let S be a subset of linear space X. the *linear span* of S is the collection of all linear combinations of functions in S. That is, the linear span of S is the collection of all functions of the form $f = \sum_{k=1}^n \alpha_k f_k$ where $\alpha_k \in \mathbb{R}$ and $f_k \in S$.

Proposition 8.9. Let E be a measurable set, $1 \le p < \infty$, and let q be the conjugate of p. Assume $\mathcal{F}$ is a subset of $L^q(E)$ whose linear span is dense in $L^q(E)$. Let $\{f_n\}$ be a bounded sequence in $L^p(E)$ and let f belong to $L^p(E)$. Then $\{f_n\} \rightharpoonup f$ in $L^p(E)$ if and only if $\lim_{n\to\infty} \left(\int_E f_n g \right) = \int_E fg$ for all $g \in \mathcal{F}$.

Note. The power of Proposition 8.9 is seen in the following two theorems.

Theorem 8.10. Let E be a nonmeasurable set and $1 \le p < \infty$, suppose $\{f_n\}$ is a bounded sequence in $L^p(E)$ and f belongs to $L^p(E)$. Then $\{f_n\} \rightharpoonup f$ in $L^p(E)$ if and only if for every measurable subset A of E we have $\lim b \to \infty \left(\int_A f_n \right) = \int_A f$. If $p > 1$ (and so $q < \infty$) it is sufficient to consider sets A of finite measure.

Theorem 8.11. Let $[a, b]$ be a closed, bounded interval and $1 < p < \infty$. Suppose $\{f_n\}$ is a bounded sequence in $L^p[a, b]$ and $f \in L^p[a, b]$. Then $\{f_n\} \rightharpoonup f$ in $L^p[a, b]$ if and only if $\lim_{n\to\infty} \left(\int_a^x f_n \right) = \int_a^x f$ for all $x \in [a, b]$. (This result is false for $p = 1$ since, as shown in Section 8.4 Problem 8.44, step functions are not dense in $L^\infty[a, b]$.)

Note. We now look at how pointwise convergence compares to weak convergence.

Example. Consider $p = 1$ and $q = \infty$. For $n \in \mathbb{N}$ define $f_n = n\chi_{(0,a/n]}$ on $[0, 1]$, and let $f \equiv 0$. Then $\{f_n\}$ converges pointwise to f (notice that it is not L^1 convergent, though). Let $g = \chi_{[0,1]} \in L^\infty[0, 1]$. Then $\left(\int_{[0,1]} f_n g\right) = \lim_{n\to\infty} \left(\int_{[0,1]} f_n\right) = 1$, but $\int_{[0,1]} fg = \int_{[0,1]} f = 0$. So by Proposition 8.6, $\{f_n\}$ does not converge weakly to f. So this is an example of a pointwise convergent bounded sequence in $L^1[0, 1]$ that is not weakly convergent in $L^1[0, 1]$. The following result shows that this situation does not occur for $1 < p < \infty$.

Theorem 8.12. Let E be a measurable set and $1 < p < \infty$. Suppose $\{f_n\}$ is a bounded sequence in $L^p(E)$ that converges pointwise a.e. on E to f. Then $\{f_n\} \rightharpoonup f$ in $L^p(E)$.

Note. The following result tells us (for $1 < p < \infty$) when weak convergence in $L^p(E)$ implies strong convergence in $L^p(E)$. Notice that it is similar to Theorem 7.7, but the hypothesis of pointwise convergence in Theorem 7.7 is replaced with a hypothesis of weak sequential convergence.

The Radon-Riesz Theorem.

Let E be a measurable set and $1 < p < \infty$. Suppose $\{f_n\} \rightharpoonup f$ in $L^p(E)$. Then $\{f_n\} \to f$ in $L^p(E)$ if and only if $\lim_{n\to\infty} \|f_n\|_p = \|f\|_p$.

Note. The text gives a proof of the Radon-Riesz Theorem for $p = 2$ (in which case $q = 2$ as well), which is surprisingly easy. The case for general p is not presented, but the text references the book *Functional Analysis* by Frigyes Riesz and Béla Sz.-Nagy (Dover Publishing, 1990; pages 78–80). Detailed notes based on this source are in Supplement. The Radón-Riesz Theorem.

Corollary 8.13. Let E be a measurable set and $1 < p < \infty$. Suppose $\{f_n\} \rightharpoonup f$ in $L^p(E)$. Then a subsequence of $\{f_n\}$ converges strongly in $L^p(E)$ to f if and only if $\|f\|_p = \liminf \|f_n\|_p$.

Note. The Radon-Riesz Theorem does not hold for $p = 1$. Let $n \in \mathbb{N}$ and define $f_n(x) = 1 + \sin(nx)$ on $I = [-\pi, \pi]$. then $\{f_n\} \rightharpoonup f$ in $L^1[\pi, \pi]$ where $f \equiv 1$ by Theorem 8.11 since

$$\begin{aligned} \lim_{n\to\infty} \left(\int_{-\pi}^{x} f_n \right) &= \lim_{n\to\infty} \int_{-\pi}^{x} (1 + \sin(nt)) = \lim_{n\to\infty} \left(t - \frac{1}{n}\cos(nt) \right) \Big|_{-\pi}^{\pi} \\ &= \lim_{n\to\infty} \left((x - \frac{1}{n}\cos(nx)) - (\pi - \frac{1}{n}\cos(n\pi)) \right) = x + \pi = \int_{-\pi}^{\pi} 1. \end{aligned}$$

Also, as above,

$$\lim_{n\to\infty} \|f_n\|_1 = \lim_{n\to\infty} \int_{-\pi}^{\pi} |1 + \sin(nt)|\, dt = 2\pi = \|f\|_1.$$

However, $\{f_n\} = \{1 + \sin(nx)\}$ does not converge to $f \equiv 1$ in $L^1[-\pi, \pi]$ since

$$\begin{aligned} \lim_{n\to\infty} \|f_n - f\|_1 &= \lim_{n\to\infty} \left(\int_{-\pi}^{\pi} |\sin(nx)|\, dx \right) = \lim_{n\to\infty} \left(2n \int_{0}^{\pi/n} \sin(nx)\, dx \right) \\ &= \lim_{n\to\infty} 2n \left(-\frac{1}{n}\cos(nx) \right) \Big|_{0}^{\pi/n} = \lim_{n\to\infty} (-2\cos\pi + 2\cos 0) = 4 \neq 0. \end{aligned}$$

So $\{f_n\} \rightharpoonup f$ in $L^1(I)$, $\lim\limits_{n\to\infty} \|f_n\|_1 = \|f\|_1$, but $\{f_n\}$ does not converge strongly to f in $L^1(I)$.