Section 8.3. Weak Sequential Compactness

Note. In this section, we define a new sort of compactness which applies in normed linear spaces. First, we resolve the Bolzano-Weierstrass Theorem in the L^p setting with the following result. This result will be used to illustrate the new compactness.

Theorem 8.14. Let E be a measurable set and $1 < p < \infty$. Then every founded sequence in $L^p(E)$ has a subsequence that converges weakly in $L^p(E)$ to a function in $L^p(E)$.

Note. Our proof of Theorem 8.14 depends on the following.

Helly's Theorem. Let X be a separable normed linear space and $\{T_n\}$ a sequence in its dual space X^* that is bounded. That is, there exists $M \geq 0$ for which $|T_n(f)| \leq M\|f\|$ for all $f \in X$, for all $n \in \mathbb{N}$. Then there is a subsequence $\{T_{n_k}\}$ of $\{T_n\}$ and $T \in X^*$ for which $\lim_{k\to\infty} T_{n_k}(f) = T(f)$ for all $f \in X$.

Note. Recall from Theorem 7.11 that $L^p(E)$ is separable for $1 \leq p < \infty$, so Helly's Theorem applies in these cases. Problem 8.36 shows that Helly's Theorem does not hold for $p = \infty$. We are now ready for the proof of Theorem 8.14.

Example. Theorem 8.14 does not hold for $p = 1$. For example, consider $L^1[0,1]$. Define $I_n = [0, 1/n]$ and $f_n = n\chi_{I_n}$ for $n \in \mathbb{N}$. Then $\{f_n\}$ is bounded since $\|f_n\|_1 = 1$ for all $n \in \mathbb{N}$. ASSUME subsequence $\{f_{n_k}\}$ converges weakly in $L^1[0,1]$ to $f \in L^1[0,1]$. For each $[c,d] \subset [0,1]$ integration against $\chi_{[c,d]}$ is a bounded linear functional on $L^1[0,1]$ (by Proposition 8.2 since $\chi_{[c,d]} \in L^\infty[0,1]$). Thus

$$\begin{aligned}
\int_c^d f &= \int_0^1 f\chi_{[c,d]} \\
&= \lim_{k\to\infty}\left(\int_0^1 f_{n_k}\chi_{[c,d]}\right) \text{ by weak convergence (Proposition 8.6)} \\
&= \lim_{k\to\infty}\left(\int_c^d f_{n_k}\right).
\end{aligned}$$

Now, for $c > 0$, $\displaystyle\int_c^d f_{n_k} = 0$ for k sufficiently large, since $f_{n_k} = n_k\chi_{[0,1/n_k]}$. So for all $0 < c < d \leq 1$, $\displaystyle\int_c^d f = \lim_{k\to\infty}\left(\int_c^d f_{n_k}\right) = 0$. Then, by Lemma 6.13, $f = 0$ a.e. on $[0,1]$. Therefore,

$$0 = \int_0^1 f = \lim_{k\to\infty}\left(\int_0^1 f_{n_k}\right) = 1$$

where the center equality holds by weak convergence (Proposition 8.6) and $g \equiv 1 \in L^\infty[0,1]$, a CONTRADICTION. This contradiction shows that the supposition of a weakly convergent subsequence is false.

Note. Although a bounded sequence in $L^1(E)$ may not have a weakly convergent subsequence (as shown in the previous example), a bounded sequence in $L^1(E)$ where $m(E) < \infty$, which is also *uniformly convergent* has a weakly convergent subsequence. The proof of this is given in a more general setting in Chapter 19 (see page 412, Theorem 19.12, the Dunford-Pettis Theorem).

Definition. A subset K of a normed linear space X is *weakly sequentially compact* in X if every sequence $\{f_n\}$ in K has subsequence that converges weakly to $f \in K$.

Note. Recall that a set of real numbers A is compact if and only if every sequence $\{a_n\} \subset A$ has a subsequence which converges to an element of A. This is true in more general settings as well. For example, it holds in metric spaces (see Theorem 9.16 on page 199). Since strong convergence (that is, convergence in $L^p(E)$) implies weak convergence, we see that the previous definition is a generalization of the idea of "regular" compactness for a set and its relationship to the behavior of sequences from the set.

Theorem 8.15. Let E be a measurable set and $1 < p < \infty$. Then $\{f \in L^p(E) \mid \|f\|_p \leq 1\}$ is weakly sequentially compact in $L^p(E)$.

Note. The set $\{f \in L^p(E) \mid \|f\|_p \leq 1\}$ is actually the closed unit ball in $L^p(E)$.

Revised: 1/14/2019