Essential Background for Real Analysis I (MATH 5210)

Note. These notes contain several definitions, theorems, and examples from Analysis I (MATH 4217/5217) which you must know for this class. Throughout, we give references to two texts used in the past in Analysis I, James R. Kirwood's *An Introduction to Analysis*, 2nd Edition (Waveland Press, 2002), Kenneth A. Ross's *Elementary Analysis: The Theory of Calculus* (Springer, Undergraduate Texts in Mathematics, 1980), and Halsey Royden and Patrick Fitzpatrick's *Real Analysis*, 4th Edition (Prentice Hall, 2010—in these notes, we refer to this book simply as "Royden").

Note. It is assumed that you recall standard proof techniques from Mathematical Reasoning (MATH 3000) and their applications to set theoretic arguments. In particular, the following.

Theorem 0.1. De Morgan's Laws. (Kirkwood, Exercise 1.1.8; Ross, Exercise 2.13.5(a); Royden, page 4)
Let $\{A_i \mid i \in I\}$ be any collection of sets. Then

$$(\cap_{i\in I} A_i)^c = \cup_{i\in I} A_i^c \text{ and } (\cup_{i\in I} A_i)^c = \cap_{i\in I} A_i^c.$$

Definition of the Real Numbers, $\mathbb{R}$

Definition 0.1. (Kirkwood, page 14; Ross, page 13; Royden, pages 7 and 8)
A field $\mathbb{F}$ is a nonempty set together with two operations $+$ and $\cdot$, called addition and multiplication, which satisfy the following axioms:

Axiom 1. If $a, b \in \mathbb{F}$ then $a + b$ and $a \cdot b$ are uniquely determined elements of $\mathbb{F}$ (i.e., $+$ and $\cdot$ are *binary operations*).

Axiom 2. If $a, b, c \in \mathbb{F}$ then $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ (i.e., $+$ and $\cdot$ are *associative*).

Axiom 3. If $a, b \in \mathbb{F}$ then $a + b = b + a$ and $a \cdot b = b \cdot a$ (i.e., $+$ and $\cdot$ are *commutative*).

Axiom 4. If $a, b, c \in \mathbb{F}$ then $a \cdot (b + c) = a \cdot b + a \cdot c$ (i.e., $\cdot$ *distributes* over $+$).

Axiom 5. There exists $0, 1 \in \mathbb{F}$ such that $0 + a = a$ and $1 \cdot a = a$ for all $a \in \mathbb{F}$.

Axiom 6. If $a \in \mathbb{F}$ then there exists $-a \in \mathbb{F}$ such that $a + (-a) = 0$.

Axiom 7. If $a \in \mathbb{F}$ $a \neq 0$, then there exists a^{-1} such that $a \cdot a^{-1} = 1$.

0 is the *additive identity*, 1 is the *multiplicative identity*, $-a$ and a^{-1} are *inverses* of a.

Example. Some examples of fields are $\mathbb{Z}_p$ for p prime, the rational numbers $\mathbb{Q}$, the extension of the algebraic numbers $\mathbb{Q}(\sqrt{2}, \sqrt{3})$, the algebraic numbers $\mathbb{A}$, the real numbers $\mathbb{R}$, and the complex numbers $\mathbb{C}$.

Definition 0.2. (Kirkwood, page 16; Ross, page 13; Royden, pages 8 and 9)
Let $\mathbb{F}$ be a field. Then $\mathbb{F}$ is an *ordered field* if it satisfies the axiom:

Axiom 8. There is $P \subset \mathbb{F}$ (called the *positive subset*) such that

(i) If $a, b \in P$ then $a + b \in P$ (closure of P under addition).

(ii) If $a, b \in P$ then $a \cdot b \in P$ (closure of P under multiplication).

(iii) If $a \in \mathbb{F}$ then exactly one of the following holds: $a \in P$, $-a \in P$, or $a = 0$ (this is *The Law of Trichotomy*).

Definition 0.3. (Kirkwood, page 17; Ross, page 13; Royden, pages 8 and 9)
Let $\mathbb{F}$ be a field and P the positive subset. We say that $a < b$ (or $b > a$) if $b - a \in P$.

Note. The field $\mathbb{Z}_p$ (p prime) is not ordered. Fields $\mathbb{Q}$, $\mathbb{Q}(\sqrt{2}, \sqrt{3})$, and $\mathbb{R}$ are ordered by the traditional "<." Field $\mathbb{C}$ is not ordered (see my notes for Complex Analysis I [MATH 5510] on Ordering the Complex Numbers).

Definition 0.4. (Kirkwood, page 17; Ross, page 20)
An *interval* in an ordered field is a set A containing at least two elements such that is $r, s \in A$ with $r < s$ and if t is an element of the field such that $r < t < s$ then $t \in A$.

Note. There are several types of intervals depending on whether they have and/or include "endpoints."

Note. In order to axiomatically define the real numbers, we need to address the concept of "completeness." It is this concept which will insure that the real numbers form a continuum and that it has "no holes."

Definition 0.5. (Kirkwood, page 25; Ross, Definition 1.4.2; Royden, page 9) Let A be a subset of an ordered field $\mathbb{F}$. If there exists $b \in \mathbb{F}$ such that $a \leq b$ for all $a \in A$, then b is an *upper bound* of A and A is said to be *bounded above.* If there exists $c \in \mathbb{F}$ such that $c \leq a$ for all $a \in A$, then c is a *lower bound* of A and A is *bounded below.* A set bounded above and below is *bounded.* A set that is not bounded is *unbounded.*

Definition 0.6. (Kirkwood, page 25; Ross, Definition 1.4.3; Royden, page 9) Let A be a subset of an ordered field $\mathbb{F}$ which is bounded above. Then $b \in \mathbb{F}$ is called a *least upper bound* (*lub* or *supremum*) of a set A if (1) b is an upper bound of A and (2) if c is an upper bound of A, then $b \leq c$.

Definition 0.7. (Kirkwood, page 25; Ross, Definition 1.4.3; Royden, page 9) Let A be a subset of an ordered field $\mathbb{F}$ which is bounded below. Then $b \in \mathbb{F}$ is called a *greatest lower bound* (*glb* or *infimum*) of a set A if (1) b is an lower bound of A and (2) if c is an lower bound of A, then $c \leq b$.

Definition 0.8. (Kirkwood, page 25; Ross, page 22; Royden, page 9) Let $\mathbb{F}$ be an ordered field. $\mathbb{F}$ is *complete* if for any nonempty set $A \subset \mathbb{F}$ that is bounded above, there is a lub of A in $\mathbb{F}$.

Example. The rationals $\mathbb{Q} = \{p/q \mid p, q \in \mathbb{Z}, q \neq 0\}$ are not complete because set $A = \{x \in \mathbb{Q} \mid x^2 < 2\}$ has an upper bound (say 2) but has no least upper bound in $\mathbb{Q}$.

Definition 0.9. (Kirkwood, page 25; Ross, Axiom 4.4; Royden, page 9)

Axiom 9. The real numbers are complete.

That is, the real numbers are a complete, ordered field.

Note. The 9 axioms of the real numbers consist of 7 Field Axioms, the Order Axiom, and the Completeness Axiom. So the real numbers are *a* complete ordered field. In fact, we can say that the real numbers are *the* complete ordered fields, since it can be shown that all complete ordered fields are isomorphic. Details on the can be found in *Which Numbers are Real?* by Micael Henle, Washington, DC: Mathematical Association of America, Inc. (2012) (see Theorem 2.3.3 of page 48).

Note. The Axiom of Completeness can be used to define exponentiation of positive real numbers to irrational powers. It is also possible to define this exponentiation in terms of the exponential function, e^x; this is the approach taken in Calculus II and by Ross (Definition 37.7). However, the exponential function is defined in Calculus and in Ross using the theory of integration where the natural logarithm function is defined in terms of integrals and the exponential function is then defined as the inverse of the natural logarithm function.

Theorem 0.2. (Kirkwood, page 26)

Let $x > 1$ be a positive real number and r a positive irrational number. Then x^r is the lub of the set $\{x^p \mid p \in \mathbb{Q}, 0 < p < r\}$. If $x \in (0,1)$, then x^r is $\dfrac{1}{(1/x)^r}$. If r is a negative irrational number, then $x^r = (1/x)^{|r|}$.

Note. We will define the measure of a set and certain Lebesgue integrals in terms of suprema and infima so the following property is important.

Theorem 0.3. Epsilon Property of Sup and Inf. (Kirkwood's Theorem 1-15; Royden, Proposition 1.19(i))

(a) Finite α is a lub of $A \subset \mathbb{R}$ if and only if

- **(i)** α is an upper bound of A, and
- **(ii)** For all $\varepsilon > 0$ there exists a number $x(\varepsilon) \in A$ such that $x(\varepsilon) > \alpha - \varepsilon$.

(b) Finite β is a glb of $A \subset \mathbb{R}$ if and only if

- **(i)** β is a lower bound of A, and
- **(ii)** For all $\varepsilon > 0$ there exists a number $x(\varepsilon) \in A$ such that $x(\varepsilon) < \beta + \varepsilon$.

Sequences of the Real Numbers

Note. You should be familiar with the definition of a sequence of real numbers and the epsilon-delta definition of the limit of a sequence from Calculus II (MATH 1920).

Definition 0.10. (Kirkwood, page 46; Ross, Definition 2.10.8, Royden, page 22)
A sequence of real numbers $\{a_n\}$ is a *Cauchy sequence* if

$$\text{for all } \varepsilon > 0, \text{ there exists } N(\varepsilon) \text{ such that}$$
$$\text{if } n, m > N(\varepsilon) \text{ then } |a_n - a_m| < \varepsilon.$$

Note. The fact that a convergent sequence is Cauchy is a simple consequence of the Triangle Inequality for real numbers ($|a + b| \le |a| + |b|$ for all $a, b \in \mathbb{R}$). The fact that a Cauchy sequence of real numbers is convergent is a consequence of the Axiom of Completeness. We then have: A sequence of real numbers is convergent if and only if it is Cauchy (see Kirkwood, Exercise 2.3.13; Ross, Theorem 2.10.11; Royden and Fitzpatrick, Theorem 1.17).

Definition 0.11. (Kirkwood, page 48; Ross, Definition 2.11.1)
Let $n_1 < n_2 < \cdots < n_k < \cdots$ be strictly increasing sequence of positive integers. Then $a_{n_1}, a_{n_2}, \ldots, a_{n_k}, \ldots$ is a *subsequence* of $\{a_n\}$ and is denoted $\{a_{n_k}\}$.

Definition 0.12. (Kirkwood, page 49; Ross, Definition 2.11.6)
L is a *subsequential limit* of $\{a_n\}$ if there is a subsequence of $\{a_n\}$ that converges to L.

Definition 0.13. (Kirkwood, page 55; Ross, Theorem 2.11.7)
Let $\{a_n\}$ be a sequence of real numbers. Then $\limsup a_n = \overline{\lim}\, a_n$ is the least upper bound of the set of subsequential limits of $\{a_n\}$, and $\liminf a_n = \underline{\lim}\, a_n$ is the greatest lower bound of the set of subsequential limits of $\{a_n\}$.

Note. Ross and Royden and Fitzpatrick have a different (but equivalent) definition of lim sup and lim inf for $\{a_n\}$ (Ross, Definition 2.10.6; Royden, page 23):

$$\limsup\{a_n\} = \lim_{n\to\infty}[\sup\{a_k \mid k \geq n\}].$$

$$\liminf\{a_n\} = \lim_{n\to\infty}[\inf\{a_k \mid k \geq n\}].$$

Note. For a given sequence $\{a_n\}$, we have that property that $\limsup a_n$ and $\liminf a_n$ are both subsequential limits (Kirkwood, Exercise 2.3.16). So we can say that $\limsup a_n$ is the greatest subsequential limit of sequence $\{a_n\}$ and $\liminf a_n$ is the least subsequential limit of sequence $\{a_n\}$ (with the obvious interpretation for $-\infty$ and ∞).

Theorem 0.4. (Kirkwood, Theorem 2-18; Ross, Exercises 2.12.4 and 2.12.5) Let $\{a_n\}$ and $\{b_n\}$ be bounded sequences of real numbers.

(a) $\overline{\lim}(a_n + b_n) \leq \overline{\lim}\, a_n + \overline{\lim}\, b_n$, and

(b) $\underline{\lim}\, a_n + \underline{\lim}\, b_n \leq \underline{\lim}(a_n + b_n)$.

Open and Closed Sets, Compact Sets, Connected Sets

Definition 0.14. (Kirkwood, pages 60 and 61; Ross, Definition 2.13.6) A set U of real numbers is said to be *open* if for all $x \in U$ there exists $\delta(x) > 0$ such that $(x - \delta(x), x + \delta(x)) \subset U$. A set A is *closed* if A^c is open.

Theorem 0.5. (Kirkwood, Theorem 3-2; Ross, Exercise 2.13.3)
The open sets satisfy:

(a) If $\{U_1, U_2, \ldots, U_n\}$ is a *finite* collection of open sets, then $\cap_{k=1}^n U_k$ is an open set.

(b) If $\{U_\alpha\}$ is *any* collection (finite, infinite, countable, or uncountable) of open sets, then $\cup_\alpha U_\alpha$ is an open set.

Theorem 0.6. (Kirkwood, Theorem 3-3; Ross, Exercise 2.13.5(b))
The closed sets satisfy:

(a) $\emptyset$ and $\mathbb{R}$ are closed.

(b) If $\{A_\alpha\}$ is *any* collection of closed sets, then $\cap_\alpha A_\alpha$ is closed.

(c) If $\{A_1, A_2, \ldots, A_n\}$ is a *finite* collection of closed sets, then $\cup_{k=1}^n A_i$ is closed

Note. We see from the previous two results that (1) a countable infinite intersection of open sets may not be open, and (2) a countable infinite union of closed sets may not be closed. In this class, we will give such sets names (G_δ and F_σ, respectively).

Note. The following is **the most important result from Analysis I** for Real Analysis I! It tells us what an open set of real numbers "looks like."

Theorem 0.7. Classification of Open Sets of Real Numbers. (Kirkwood, Theorem 3-5; Ross, Exercise 3.13.7)
A set of real numbers is open if and only if it is a countable union of disjoint open intervals.

Note. A largely self-contained proof of Theorem 0.7 is in my online notes for Analysis 1 on A Classification of Open Sets of Real Numbers.

Definition 0.15. (Kirkwood, pages 65 and 66; Ross, Definition 2.13.11)
Let $A \subset \mathbb{R}$. The collection of sets $\{I_\alpha\}$ is said to be a *cover* of A if $A \subset \cup_\alpha I_\alpha$. If each I_α is open, then the collection is called an *open cover* of A. If the cardinality of $\{I_\alpha\}$ is finite, then the collection is a *finite cover*. A set of real numbers is *compact* if every open cover of the set has a finite subcover.

Theorem 0.8. Heine-Borel Theorem. (Kirkwood, Theorems 3-10 and 3-11; Ross, Theorem 2.13.12)
A set of real numbers is compact if and only if it is closed and bounded.

Definition 0.16. (Kirkwood, page 69; Ross, Definition 3.22.1)
A *separation* of set A is two open sets U and V such that:

(i) $U \cap V = \emptyset$.

(ii) $U \cap A \neq \emptyset$ and $V \cap A \neq \emptyset$.

(iii) $(U \cap A) \mathbin{\dot\cup} (V \cap A) = A$.

A set A is *connected* if there does not exist a separation of A.

Theorem 0.9. (Kirkwood, Theorem 3-14)

A set of real numbers is connected if and only if it is an interval or a singleton.

Cardinalities: Cantor's Theorem and Alephs

Note. Cardinalities of sets play a large role in analysis. We now touch lightly on a few important concepts.

Definition 0.17. (Kirkwood, page 30; Royden, page 5)

Two sets A and B are said to have the *same cardinality* if there is a one-to-one and onto function from A to B.

Definition 0.18. (Kirkwood, page 20; Royden, page 13)

A set S is said to be *finite with cardinality* n if there is a one-to-one and onto function from S to $\{1, 2, \ldots, n\}$ where $n \in \mathbb{N}$. The empty set is *finite with cardinality* 0. Sets that do not have finite cardinality are *infinite sets.* A set S is *countable* if it has the same cardinality as some subset of $\mathbb{N}$. Sets that are not countable are said to be *uncountable.*

Theorem 0.10. (Kirkwood, Theorem 1-19; Royden, Corollary 1.6)

The union of a countable collection of countable sets is countable.

Theorem 0.11. (Kirkwood, Theorem 1-20; Royden, Theorem 1.7)
The real numbers in $(0,1)$ form an uncountable set.

Note. The technique of proof in Theorem 0.11 is called the "Cantor diagonalization argument." Since $\tan(\pi x - \pi/2)$ is a one to one and onto mapping from $(0,1)$ to $\mathbb{R}$, it follows by Definition 0.17 that the cardinality of $(0,1)$ is the same as the cardinality of $\mathbb{R}$ (in fact, every interval has the same cardinality as $\mathbb{R}$).

Definition 0.19. (Kirkwood, page 33)
A *cardinal number* is associated with a set. Two sets share the same cardinal number if they are of the same cardinality. The cardinal number of set X is denoted $|X|$. We order the cardinal numbers with the following:

(i) If X and Y are sets and there is a one to one function from X into Y, then the cardinal number of X is no larger than the cardinal number of Y, denoted $|X| \le |Y|$ or $|Y| \ge |X|$.

(ii) If (i) holds, and if there is no onto function from X to Y, then the cardinal number of Y is strictly larger than the cardinal number of X, denoted $|X| < |Y|$ or $|Y| > |X|$.

(We interpret the idea of the cardinal number of Y being larger than the cardinal number of X as meaning that Y has "more elements" than X.)

Theorem 0.12. Cantor's Theorem. (Kirkwood, Theorem 1-21)
The cardinal number of the power set of X, $\mathcal{P}(X)$, is strictly larger than the cardinal number of X, $|X| < |\mathcal{P}(X)|$.

Note. For a finite set A with $|A| = n$, it is easy to show that $|\mathcal{P}(A)| = 2^n$ (by induction, say). Based on this fact, it is a common notation to denote $|\mathcal{P}(A)| = 2^{|A|}$. It can be shown that $|\mathbb{N}|$ is the smallest infinite cardinal number. This cardinality is denoted "aleph naught": $\aleph_0 = |\mathbb{N}|$. So the cardinality of any countable set is $\aleph_0$. By Cantor's Theorem, we have $\aleph_0 = |\mathbb{N}| < |\mathcal{P}(\mathbb{N})| = 2^{\aleph_0}$.

Note. The real numbers, $\mathbb{R}$, are often called "the continuum" and the cardinality of the continuum is denoted $|\mathbb{R}| = c$. It can be shown that $c = |\mathbb{R}| = |\mathcal{P}(\mathbb{N})| = 2^{\aleph_0}$ (see Section 6.2 of Hrbacek and Jech, *Introduction to Set Theory*, 2nd Edition Revised and Expanded, in Pure and Applied Mathematics, A Series of Monographs and Textbooks, Marcel Dekker (1984)). Since $\aleph_0 < c$ it is natural to ask if there is a cardinal number β such that $\aleph_0 < \beta < c$. This is equivalent to the existence of a set B where $\mathbb{N} \subset B \subset \mathbb{R}$ where $|\mathbb{N}| < |B| < |\mathbb{R}|$. Unfortunately, under the standard axioms of set theory, the ZFC axioms, the claim of the existence of such a set is "undecidable" (a concept introduced by Kurt Gödel [1906–1978]).

Note. In 1900, David Hilbert (1862–1943) gave a list of 23 unsolved problems at the International Congress of Mathematics in Paris and the first problem on the list was the "continuum problem." The Continuum Hypothesis states that there is no such set B and no such cardinal number β. In 1939, Kurt Gödel proved that the Continuum Hypothesis is *consistent* with the ZFC axioms of set theory (in *The Consistency of the Continuum-Hypothesis*, Princeton University Press (1940)). In 1963, Paul Cohen (1934–2007) proved that the Continuum Hypothesis is *independent* of the ZFC axioms (in "The Independence of the Continuum Hypothesis," *Proceedings of the National Academy of Sciences of the United States of America* **50**(6): 1144–1148 and "The Independence of the Continuum Hypothesis II," ibid, **51**(1), 105–110). Cohen was awarded the Fields Medal in 1966 for his proof. This is why the Continuum Hypothesis is said to be undecidable within ZFC set theory. Under the assumption of the Continuum Hypothesis, we denote $\aleph_1 = 2^{\aleph_0} = c$ (strictly speaking, $\aleph_1$ has a definition based on the study of ordinal numbers, but we omit these details).

Note. So under the assumption of the Continuum Hypothesis, we have the notation: $\aleph_0 = |\mathbb{N}|$, $\aleph_1 = |\mathbb{R}|$, and $\aleph_2 = |\mathcal{P}(\mathbb{R})|$. Real Analysis I is (crudely put) the study of $\mathcal{P}(\mathbb{R})$ and its subsets. As a consequence, these are the cardinal numbers of interest for this class.

Riemannian Integration

Note. You also need a knowledge of the development of Riemann integration. This is contained in Chapter 6 of Kirkwood and Chapter 6 of Ross. The material is also introduced in Royden's Section 4.1. We will cover these ideas in a supplement to this class on The Riemann-Lebesgue Theorem. The supplement gives the definition of a Riemann integral, results concerning uniform convergence of sequences of functions and Riemann integrability, and necessary and sufficient conditions for a bounded function to be Riemann integrable on a closed and bounded interval (the "Riemann-Lebesgue Theorem"). In addition, the concept of a set of real numbers having "measure zero" is defined.

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