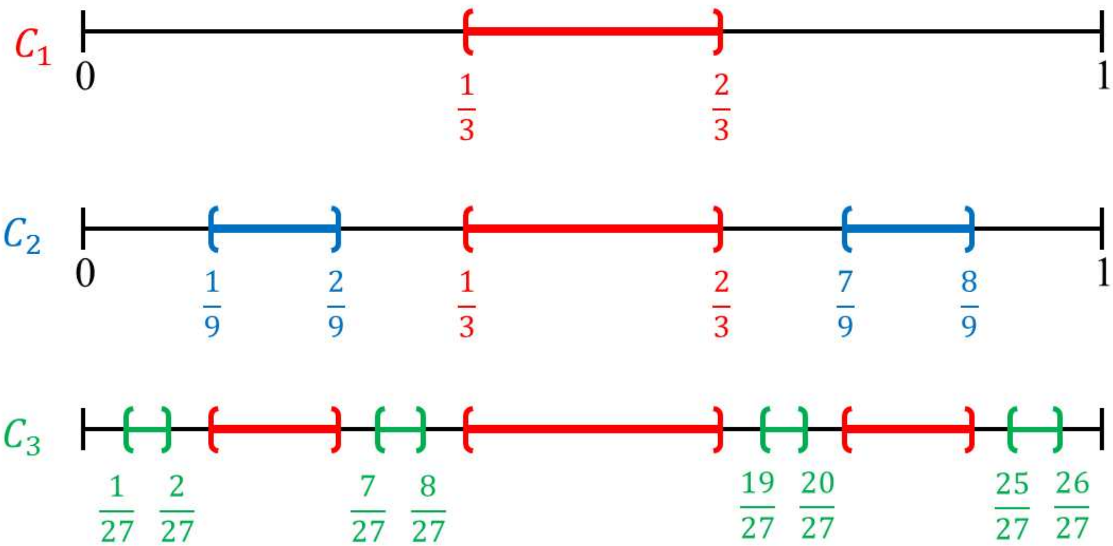
Supplement. The Cardinality of the Set of Lebesgue Measurable Sets

Note. In Section 2.6, we used the Axiom of Choice to "construct" a single nonmeasurable set. A survey of measure theory textbooks will reveal that this example is ubiquitous. As shown by Robert Solovay in 1964, we cannot construct a nonmeasureable set without the Axiom of Choice ["A Model of Set Theory in which Every Set of Reals is Lebesgue Measurable, *Annals of Mathematics* **92**(1970), 1–56]. This might lead us to believe that there are very few nonmeasurable sets, since we seem to have only one technique by which to construct them! In this supplement to the Real Analysis 1 (MATH 5210) notes, we explore the questions: (1) $|\mathcal{M}| =?$ and (2) $|\mathcal{P}(\mathbb{R}) \setminus \mathcal{M}| =?$, where $\mathcal{M}$ is the σ-algebra of Lebesgue measurable sets.

Recall. The Cantor set is constructed by starting with the interval $[0,1]$ and removing "middle thirds." These middle thirds are:

$$\begin{aligned}
C_1 &= \left(\tfrac{1}{3}, \tfrac{2}{3}\right) \\
C_2 &= \left(\tfrac{1}{9}, \tfrac{2}{9}\right) \cup \left(\tfrac{7}{9}, \tfrac{8}{9}\right) \\
C_3 &= \left(\tfrac{1}{27}, \tfrac{2}{27}\right) \cup \left(\tfrac{7}{27}, \tfrac{8}{27}\right) \cup \left(\tfrac{19}{27}, \tfrac{20}{27}\right) \cup \left(\tfrac{25}{27}, \tfrac{26}{27}\right) \\
C_4 &= \left(\tfrac{1}{81}, \tfrac{2}{81}\right) \cup \left(\tfrac{7}{81}, \tfrac{8}{81}\right) \cup \left(\tfrac{19}{81}, \tfrac{20}{81}\right) \cup \left(\tfrac{25}{81}, \tfrac{26}{81}\right) \cup \left(\tfrac{55}{81}, \tfrac{56}{81}\right) \cup \left(\tfrac{61}{81}, \tfrac{62}{81}\right) \cup \left(\tfrac{73}{81}, \tfrac{74}{81}\right) \cup \left(\tfrac{79}{81}, \tfrac{80}{81}\right) \\
&\vdots \\
C_n &= 2^{n-1} \text{ intervals of total length } \tfrac{2^{n-1}}{3^n} \\
&\vdots
\end{aligned}$$

Note. We can illustrate the first three middle thirds pictorially as (not to scale):



Definition. The Cantor Set C is the set $C = [0,1] \setminus (\cup_{k=1}^{\infty} C_k)$ where the sets C_k are defined above.

Note. The Cantor set is closed, so it is measurable. We have $[0,1] = C \cup_{k=1}^{\infty} C_k$, so by countable additivity,

$$1 = m([0,1]) = m\left(C \cup_{k=1}^{\infty} C_k\right) = m(C) + m\left(\cup_{k=1}^{\infty} C_k\right) = m(C) + \sum_{k=1}^{\infty} m(C_k)$$

$$= m(C) + \sum_{k=1}^{\infty} \frac{2^{k-1}}{3^k} = m(C) + \frac{1}{3}\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^{k-1} = m(C) + \frac{1}{3}\left(\frac{1}{1-2/3}\right) = m(C) + 1.$$

Therefore, $m(C) = 0$.

Note. The Cantor set contains all elements of $[0,1]$ which *can* be written in base 3 without using any 1's. This may require the use of an infinite number of 2's, however:

$$\begin{aligned}
\frac{2}{3} &= 2\left(\frac{1}{3}\right) = 2(3^{-1}) = (0.2)_3 \\
\frac{20}{27} &= 2\left(\frac{1}{3}\right) + 0\left(\frac{1}{9}\right) + 2\left(\frac{1}{27}\right) \\
&= 2(3^{-1}) + 0(3^{-2}) + 2(3^{-3}) = (0.202)_3 \\
\frac{1}{3} &= 1\left(\frac{1}{3}\right) = 1(3^{-1}) = (0.1)_3 = (0.0222\cdots)_3 = (0.0\overline{2})_3.
\end{aligned}$$

In fact, any number with base three ("ternary") representation which starts "0.1" lies in C_1. Any number starting "0.01" lies in the first part of C_2, and any number starting "0.21" lies in the second part of C_2. Similarly, the four parts of C_3 include numbers with ternary representations which start "0.001," "0.021," "0.201," and "0.221," respectively. So there is a natural one to one onto mapping from the Cantor set to all sequences of 0's and 2's. Since the set of all sequences of elements of A, where $|A| \geq 2$, is uncountable, then the Cantor set is uncountable. Therefore, the Cantor set is an example of a set of measure 0 which is uncountable.

Note. We can explicitly map the Cantor set onto $[0,1]$ as follows. For $x = (0.(t_1)(t_2)(t_3)\ldots)_3$ an element of C with ternary representation as given (using only 0's and 2's), define

$$f(x) = \left(0.\left(\frac{t_1}{2}\right)\left(\frac{t_2}{2}\right)\left(\frac{t_3}{2}\right)\cdots\right)_2.$$

That is, turn each of the 2's in the ternary representation into 1's and consider these as binary representations. Notice that this is not a one to one mapping since, for example, $f(1/3) = f(2/3) = 1/2$.

Theorem A. The cardinality of the σ-algebra of measurable sets is the same as the cardinality of the power set of the reals: $|\mathcal{M}| = |\mathcal{P}(\mathbb{R})| = \aleph_2$ (using notation which assumes the continuum hypothesis).

Proof. Since the Cantor set is uncountable, $|C| = \aleph_1$. So the power set of C satisfies $|\mathcal{P}(C)| = \aleph_2$. Since the Cantor set has measure zero, then each subset of C has measure zero and is therefore measurable (Proposition 2.4 of Royden and Fitzpatrick). So $\mathcal{P}(C) \subset \mathcal{M} \subset \mathcal{P}(\mathbb{R})$ and therefore $|\mathcal{M}| = \aleph_2$. ∎

Note. Of course Theorem A is "good news" in that we see that there are *lots* of measurable sets! Recall that we already knew that the Borel sets are all measurable (Theorem 2.9), but there are only $\aleph_1$ Borel sets [Corollary 4.5.3 of Inder Rana's *An Introduction to Measure and Integration* (2nd Edition, A,M.S. Graduate Studies in Mathematics Volume 45, 2002)]. We now turn to our original question and explore the "bad news."

Theorem B. The cardinality of the set of nonmeasurable sets is the same as the cardinality of the power set of the reals: $|\mathcal{P}(\mathbb{R}) \setminus \mathcal{M}| = |\mathcal{P}(\mathbb{R})|$ (using notation which assumes the continuum hypothesis).

Proof. We can generate a nonmeasurable set for each subset K of the Cantor set. Let P denote the nonmeasurable set constructed in Section 2.6. Then $\{P \cup (K+1) \mid K \subset C\}$ is a set of nonmeasurable sets and is of cardinality $\aleph_2$. ∎

Note. We see that things are as bad as they could be! Unfortunately, the cardinality of the nonmeasurable sets is the same as the cardinality of the measurable sets (and the same as the cardinality of the power set of the reals)!

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