Supplement. The Dirac Delta Function, A Cautionary Tale

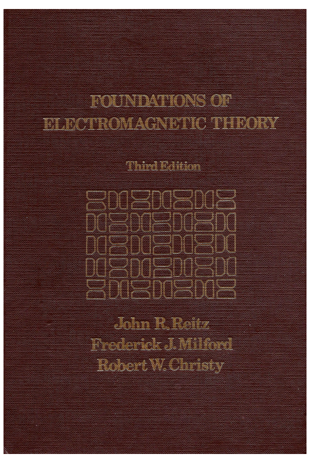
Note. In the study of charge distributions in electricity and magnetism, when considering point charges it is common to introduce the "Dirac delta function," $\delta(x)$. This function is defined to be extended real valued with the following properties:

$$\delta(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(x)\,dx = 1.$$

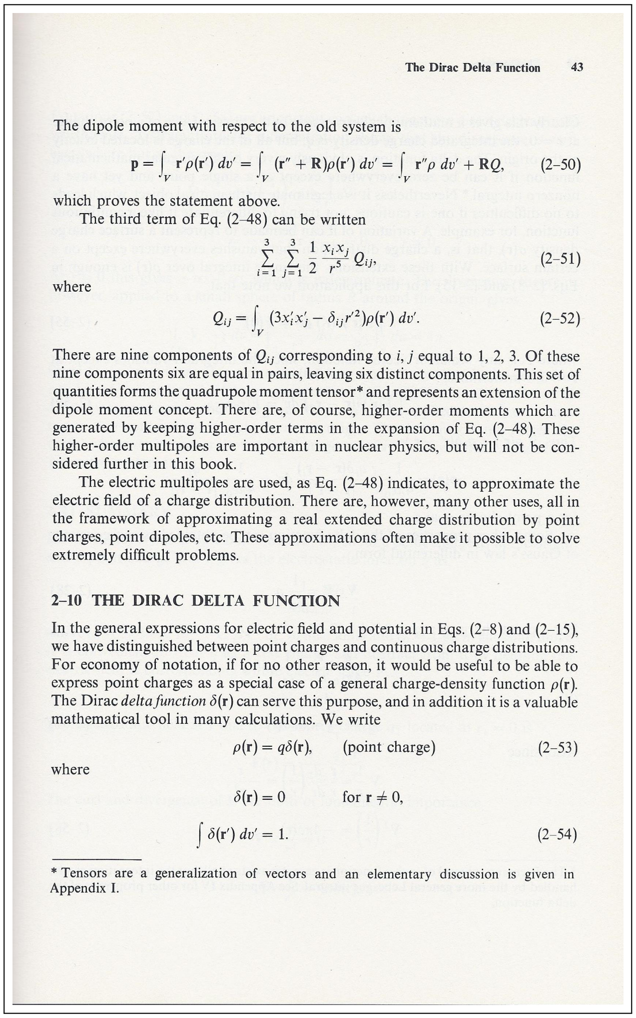
However, this is very misleading! First, Riemann integrals do not address extended real valued functions. Second, such a function would violate Proposition 4.9 of Royden and Fitzpatrick's *Real Analysis*, 4th Edition:

> **Proposition 4.9.** Let f be a nonnegative measurable (extended real valued) function on set E. Then the Lebesgue integral $\int_E f = 0$ if and only if $f = 0$ a.e. on E.

Note. In this supplement, we refer to the junior-level undergraduate text *Foundations of Electromagnetic Theory*, 3rd Edition by John Reitz, Frederick Milford, Robert Christy, Addison-Wesley (1979). We reveal some errors in it's presentation of $\delta(x)$ and explore an alternative approach to "get the math right."



Note. Reitz, Milford, and Christy introduce the Dirac delta function on pages 43 and 44.



The Dirac Delta Function 43

The dipole moment with respect to the old system is

$$\mathbf{p} = \int_V \mathbf{r}'\rho(\mathbf{r}')\,dv' = \int_V (\mathbf{r}'' + \mathbf{R})\rho(\mathbf{r}')\,dv' = \int_V \mathbf{r}''\rho\,dv' + \mathbf{R}Q, \qquad (2\text{–}50)$$

which proves the statement above.

The third term of Eq. (2–48) can be written

$$\sum_{i=1}^{3}\sum_{j=1}^{3} \frac{1}{2}\frac{x_i x_j}{r^5} Q_{ij}, \qquad (2\text{–}51)$$

where

$$Q_{ij} = \int_V (3x_i'x_j' - \delta_{ij}r'^2)\rho(\mathbf{r}')\,dv'. \qquad (2\text{–}52)$$

There are nine components of Q_{ij} corresponding to i, j equal to 1, 2, 3. Of these nine components six are equal in pairs, leaving six distinct components. This set of quantities forms the quadrupole moment tensor* and represents an extension of the dipole moment concept. There are, of course, higher-order moments which are generated by keeping higher-order terms in the expansion of Eq. (2–48). These higher-order multipoles are important in nuclear physics, but will not be considered further in this book.

The electric multipoles are used, as Eq. (2–48) indicates, to approximate the electric field of a charge distribution. There are, however, many other uses, all in the framework of approximating a real extended charge distribution by point charges, point dipoles, etc. These approximations often make it possible to solve extremely difficult problems.

2–10 THE DIRAC DELTA FUNCTION

In the general expressions for electric field and potential in Eqs. (2–8) and (2–15), we have distinguished between point charges and continuous charge distributions. For economy of notation, if for no other reason, it would be useful to be able to express point charges as a special case of a general charge-density function $\rho(\mathbf{r})$. The Dirac *delta function* $\delta(\mathbf{r})$ can serve this purpose, and in addition it is a valuable mathematical tool in many calculations. We write

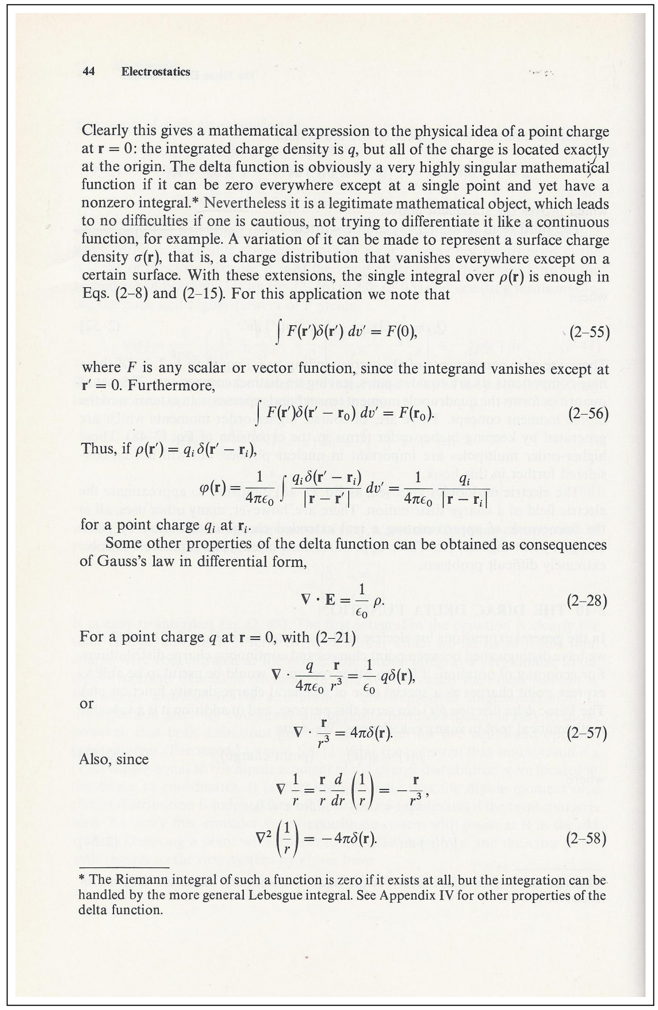
$$\rho(\mathbf{r}) = q\delta(\mathbf{r}), \qquad \text{(point charge)} \qquad (2\text{–}53)$$

where

$$\delta(\mathbf{r}) = 0 \qquad \text{for } \mathbf{r} \neq 0,$$

$$\int \delta(\mathbf{r}')\,dv' = 1. \qquad (2\text{–}54)$$

* Tensors are a generalization of vectors and an elementary discussion is given in Appendix I.



44 Electrostatics

Clearly this gives a mathematical expression to the physical idea of a point charge at $\mathbf{r} = 0$: the integrated charge density is q, but all of the charge is located exactly at the origin. The delta function is obviously a very highly singular mathematical function if it can be zero everywhere except at a single point and yet have a nonzero integral.* Nevertheless it is a legitimate mathematical object, which leads to no difficulties if one is cautious, not trying to differentiate it like a continuous function, for example. A variation of it can be made to represent a surface charge density $\sigma(\mathbf{r})$, that is, a charge distribution that vanishes everywhere except on a certain surface. With these extensions, the single integral over $\rho(\mathbf{r})$ is enough in Eqs. (2–8) and (2–15). For this application we note that

$$\int F(\mathbf{r}')\delta(\mathbf{r}')\,dv' = F(0), \qquad (2\text{–}55)$$

where F is any scalar or vector function, since the integrand vanishes except at $\mathbf{r}' = 0$. Furthermore,

$$\int F(\mathbf{r}')\delta(\mathbf{r}' - \mathbf{r}_0)\,dv' = F(\mathbf{r}_0). \qquad (2\text{–}56)$$

Thus, if $\rho(\mathbf{r}') = q_i\,\delta(\mathbf{r}' - \mathbf{r}_i)$,

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{q_i\delta(\mathbf{r}' - \mathbf{r}_i)}{|\mathbf{r} - \mathbf{r}'|}\,dv' = \frac{1}{4\pi\epsilon_0}\frac{q_i}{|\mathbf{r} - \mathbf{r}_i|}$$

for a point charge q_i at $\mathbf{r}_i$.

Some other properties of the delta function can be obtained as consequences of Gauss's law in differential form,

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho. \qquad (2\text{–}28)$$

For a point charge q at $\mathbf{r} = 0$, with (2–21)

$$\nabla \cdot \frac{q}{4\pi\epsilon_0}\frac{\mathbf{r}}{r^3} = \frac{1}{\epsilon_0}q\delta(\mathbf{r}),$$

or

$$\nabla \cdot \frac{\mathbf{r}}{r^3} = 4\pi\delta(\mathbf{r}). \qquad (2\text{–}57)$$

Also, since

$$\nabla\frac{1}{r} = \frac{\mathbf{r}}{r}\frac{d}{dr}\left(\frac{1}{r}\right) = -\frac{\mathbf{r}}{r^3},$$

$$\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta(\mathbf{r}). \qquad (2\text{–}58)$$

* The Riemann integral of such a function is zero if it exists at all, but the integration can be handled by the more general Lebesgue integral. See Appendix IV for other properties of the delta function.

Note. The motivation for $\delta(x)$ is given on page 43. On page 44, they comment that "...it is a legitimate mathematical object, which leads to no difficulties if one is cautious...." In the footnote, they falsely state: "The Riemann integral of such a function is zero if it exists at all, but the integration can be handled by the more general Lebesgue integral." Well, the Lebesgue integral does "handle" the integration, but it gives zero as the integral value.

Definition. Reitz, Milford, and Christy give more details in their Appendix IV, and the notation there is consistent with that used above in these notes (as opposed to a function of a vector variable as stated on their pages 43 and 44).

APPENDIX IV Dirac Delta Function

In one dimension, the delta function is defined by

$$\delta(x) = 0 \text{ for } x \neq 0,$$

$$\int_{-\infty}^{+\infty} \delta(x)\, dx = 1. \qquad \text{(IV–1)}$$

This can be understood in an elementary way as the limit of an ordinary continuous function. For example, the Gaussian function

$$\frac{1}{\sqrt{\pi}\,\epsilon} e^{-x^2/\epsilon^2}$$

has a peak at $x = 0$, which is higher and narrower the smaller is ϵ, and its integral from $-\infty$ to $+\infty$ is 1 for any value of ϵ. Thus we could define

$$\delta(x) = \lim_{\epsilon \to 0} \frac{1}{\sqrt{\pi}\,\epsilon} e^{-x^2/\epsilon^2}. \qquad \text{(IV–2)}$$

Another useful representation is

$$\delta(x) = \lim_{\epsilon \to 0} \frac{\epsilon}{\pi} \frac{\sin^2 x/\epsilon}{x^2}. \qquad \text{(IV–3)}$$

As a third example, we may use the Lorentzian function introduced in Chapter 19:

$$\delta(x) = \lim_{\epsilon \to 0} \frac{\epsilon}{\pi} \frac{1}{x^2 + \epsilon^2}. \qquad \text{(IV–4)}$$

An important application of the delta function involves the Fourier theorem. If a function $f(x)$ is expressed by a Fourier integral,

$$f(x) = \int_{-\infty}^{+\infty} g(k) e^{ikx}\, dx,$$

516

Dirac Delta Function 517

the theorem gives the Fourier transform

$$g(k) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(x) e^{-ikx}\, dx.$$

For $f(x) = \delta(x)$, we find $g(k) = 1/2\pi$ (see below), so that

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ikx}\, dk. \qquad \text{(IV–5)}$$

Thus the Fourier transform of the delta function is a constant function, and conversely. Taking the real part of (IV–5) gives

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \cos kx\, dk. \qquad \text{(IV–6)}$$

Equations (IV–5) and (IV–6) may be considered as other representations of the delta function.

The mean-value theorem of calculus gives

$$\int_{-a}^{+a} F(x) f(x)\, dx \cong F(0) \int_{-a}^{+a} f(x)\, dx,$$

so that as $f(x) \to \delta(x)$

$$\int F(x)\, \delta(x)\, dx = F(0). \qquad \text{(IV–7)}$$

In all of the above equations one can replace the variable x by $(x - x_0)$; in particular,

$$\int F(x)\, \delta(x - x_0)\, dx = F(x_0). \qquad \text{(IV–8)}$$

In any equation involving the delta function, the domain of integration can be reduced to any interval containing the point where the argument of the delta function vanishes.

An extension to three dimensions can be achieved by writing

$$\delta(\mathbf{r})\, dv = \delta(x)\, \delta(y)\, \delta(z)\, dx\, dy\, dz. \qquad \text{(IV–9)}$$

In Chapter 2 we found that

$$\delta(\mathbf{r}) = \nabla^2 \left(\frac{-1}{4\pi r} \right). \qquad \text{(IV–10)}$$

Note. In equations IV-2, IV-3, and IV-4 they write $\delta(x)$ as the limit of three different functions. In equation IV-2, they state $\delta(x) = \lim_{\varepsilon \to 0} \frac{1}{\sqrt{\pi}\varepsilon} e^{-x^2/\varepsilon^2}$ (a normal distribution with mean $\mu = 0$ and standard deviation $\sigma = \varepsilon/\sqrt{2}$). Since this is a normal distribution, $\int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}\varepsilon} e^{-x^2/\varepsilon^2}\, dx = 1$. As $\varepsilon \to 0$, the distribution gets very narrow with a large spike at $x = 0$. If we replace ε with $1/n$ where $n \in \mathbb{N}$, then we can create a *sequence* of functions $f_n = \frac{n}{\sqrt{\pi}} e^{-(nx)^2}$ where $\lim_{n \to \infty} f_n(x) = \delta(x)$.

Note. Recall that from Royden and Fitzpatrick:

Monotone Convergence Theorem. Let $\{f_n\}$ be an increasing sequence of nonnegative measurable functions on set E. If $\{f_n\} \to f$ pointwise a.e. on E, then

$$\lim_{n\to\infty} \left(\int_E f_n \right) = \int_E \left(\lim_{n\to\infty} f_n \right) = \int_E f.$$

In the previous note, we have $f_n = \frac{n}{\sqrt{\pi}} e^{-(nx)^2}$ and $\lim_{n\to\infty} f_n(x) = \delta(x)$, but the convergence is not monotone increasing, so the Monotone Convergence Theorem does not apply to give that the integral $\int_{-\infty}^{\infty} \delta(x)\, dx$ is 1. It is clear that Reitz, Milford, and Christy are trying to use the values of the integrals of the normal distributions to justify the claim that the integral of $\delta(x)$ should be 1, but this is not the case.

Note. Recall that from Royden and Fitzpatrick:

Fatou's Lemma. Let $\{f_n\}$ be a sequence of nonnegative measurable functions on set E. If $\{f_n\} \to f$ pointwise a.e. on E, then

$$\int_E f \le \liminf \left(\int_E f_n \right).$$

Now we can apply Fatou's Lemma to the sequence of normal distributions above, since the convergence to $\delta(x)$ is pointwise and all functions here are nonnegative. But this simply allows us to conclude that

$$\int_{-\infty}^{\infty} \delta(x)\, dx \le \int_{-\infty}^{\infty} \frac{n}{\sqrt{\pi}} e^{-(nx)^2}\, dx = 1,$$

still not an equality of 1, but merely an upper bound of 1.

Note. Reitz, Milford, and Christy's "no difficulties if one is cautious" comment is correct, with the right kind of caution! In Analysis 2 (MATH 4227/5227) Riemann-Stieltjes integrals are introduced (see my online class notes on 6-3. The Riemann-Stieltjes Integral; in particular, see Theorem 6-26). One way to resolve the desired properties of the Dirac delta function, is to use Riemann-Stieltjes integrals. You should review these notes before reading the next note.

Note. Let

$$g(x) = \begin{cases} 0 & \text{for } x \in (-\infty, 0) \\ 1 & \text{for } x \in [0, \infty). \end{cases}$$

Then the derivative of g is 0 if $n \neq 0$. The definition of limit gives that the derivative of g is ∞ at 0. Also, we have the Riemann-Stieltjes integral

$$\int_{-\infty}^{\infty} dg = \int_{-\infty}^{0} dg + (1)\left[\lim_{x \downarrow 0} g(x) - \lim_{x \uparrow 0} g(x)\right] + \int_{0}^{\infty} dg = 0 + (1)[1] + 0 = 1.$$

So the derivative of $g(x)$ has the values of $\delta(x)$ given above and the Riemann-Stieltjes integral of $f(x) = 1$ with respect to g satisfies the integral property of $\delta(x)$ given above. If f is a function defined on all of $\mathbb{R}$, then we can use the Riemann-Stieltjes integral to determine the value of f at a specific point (say $x = x_0$):

$$\int_{-\infty}^{\infty} f(x)\, dg(x - x_0) = f(x_0)\left[\lim_{x \downarrow x_0} g(x - x_0) - \lim_{x \uparrow x_0} g(x - x_0)\right] = f(x_0)[1] = f(x_0).$$

Note. Another, more sophisticated solution involves the *Dirac measure concentrated at* x_0, denoted δ_{x_0}. For a σ-algebra $\mathcal{M}$ of subsets of a set X and a point x_0 belonging to X, the measure of 1 is assigned to a set in $\mathcal{M}$ that contains x_0, and a measure of 0 is assigned to a set that does not contain x_0. We then have for measurable function f on X that $\int_X f\, d\delta_{x_0} = f(x_0)$, as desired. This is explored in Problem 18.26 (in 18.2. Integration of Nonnegative Measurable Functions) and Problem 18.27(ii) (in 18.3. Integration of General Measurable Functions).

Note. The Dirac delta function is named for Paul A. M. Dirac (August 8, 1902–October 20, 1984). Another approach is to treat $\delta(x)$ not as a function, but as a *distribution* (or a generalized function). It is therefore more accurately called the "Dirac delta distribution." A classical work on the theory of distributions is I. M. Gel'fand and G. E. Shilov's *Generalized Functions*, Volumes 15, Academic Press, (1966-1968).

Revised: 11/9/2020